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The Poincare Polynomial of an Arrangement with the Trio Separation Property

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Abstract

An arrangement of hyperplanes with a modular element in its intersection lattice has a Poincarè polynomial which factors; this was proven by Stanley in the setting of geometric lattices. This note proves a factorization in the setting of hyperplane arrangements under two conditions which imply a modular element. Two well known reflection arrangements serve as motivation and their Poincarè polynomials are computed using the main theorem of this note.

Background and Notation

Definition 1.1. Let $\mathbb{F}$ be a field. A hyperplane is an affine subspace of codimension one in $\mathbb{F}^\ell$. A hyperplane arrangement in $\mathbb{F}^\ell$ is a finite collection of hyperplanes in $\mathbb{F}^\ell$, written $\mathcal{A}=\{H_1,....,H_n\}$. The cardinality of $\mathcal{A}$ is n and is denoted $|\mathcal{A}|$.

Denition 1.2. Let $\mathcal{A}$ be an arrangement of hyperplanes in $V=\mathbb{F}^\ell$. We define the partially ordered set $L(\mathcal{A})$ with objects given by $\bigcap_{H\in B} H$ for $B\subseteq\mathcal{A}$ and $\cap_{H\in B} H\neq\emptyset$; order the objects of $L(\mathcal{A})$ opposite to inclusion. Notice $\emptyset\subseteq\mathcal{A}$ gives $V\in L(\mathcal{A})$ with $V\leq X$ for all $X\in L(\mathcal{A})$. For $X\in L(\mathcal{A})$, We define $rank(X):=co\dim X$. We define $rank(\mathcal{A}):=\max_{X\in L(\mathcal{A})} rank(X)$.

Definition 1.3. Let $\mathcal{A}$ be an arrangement. If $B\subseteq\mathcal{A}$ is a subset, then B is called a subarrangement. For $X\in L(\mathcal{A})$ we define a subarrangement A_X of $\mathcal{A}$ by $A_X:=\{H\in\mathcal{A}:X\subset H\}$. Define an arrangement $\mathcal{A}^X$ in X via $\mathcal{A}^X=\{X\cap H: H\in\mathcal{A}\setminus\mathcal{A}_X$ and $X\cap H\neq\emptyset\}$.

Definition 1.4. Let $\mathcal{A}=\{H_1,....,H_n\}$ be a hyperplane arrangement in $V=\mathbb{F}^\ell$ for some field $\mathbb{F}$. We fix an order on $\mathcal{A}$; that is, for hyperplanes H_i and H_j in $\mathcal{A}$, we have $H_i < H_j$ if and only if $i < j$.

Let K be a commutative ring. Let E_1 be the linear space over K on n generators. Let E($\mathcal{A}$) := $\Lambda(E_1)$ be the exterior algebra on E_1. We have $E(\mathcal{A})=\oplus_{p\geq 0}E_p$ is a graded algebra over K. The standard K-basis for E_p is given by

$$\{e_{i_1}...e_{i_p}:1\leq i_1<...<i_p\leq p\}.$$

Any ordered subset $S=\{H_{i_1},....,H_{i_p}\}$ of $\mathcal{A}$ corresponds to an element $e_S:=e_{i_1}...e_{i_p}$ in $E(\mathcal{A})$.

Definition 1.5. We define the map $\partial: E(\mathcal{A})\to E(\mathcal{A})$ via the usual differential. That is,

$$\partial(1):=0,$$

$$\partial(e_i):=1, \text{ and for } p\geq 2,$$

$$\partial(e_{i_1}...e_{i_p}):=\sum_{k=1}^{p}(-1)^{k-1}e_{i_1}...\hat{e}_{i_k}...e_{i_p}$$

Definition 1.6. We define $I(\mathcal{A})$ to be the ideal of $E(\mathcal{A})$ which is generated by

$$\{\partial(e_S): S \text{ is dependent}\}\cup\{e_S:\cap S=\emptyset\}.$$

Definition 1.7. The Orlik-Solomon algebra, $A(\mathcal{A})$, is defined as

$$A(A):E(A)/I(A).$$

Let $\pi: E(\mathcal{A})\to A(\mathcal{A})$ be the canonical projection. We write a_S to represent the image of e_S under π.

We define the Orlik-Solomon algebra and a linear basis for this algebra, referred to as the broken circuit basis; see Chapter 3 in [3].

Let $\mathcal{A}=\{H_1,....,H_n\}$ be a hyperplane arrangement in $V=\mathbb{F}^\ell$ for some field $\mathbb{F}$. For each $H_i\in\mathcal{A}$, we fix an affine functional α_i with Ker $\alpha_i = H_i$. We fix an order on $\mathcal{A}$; that is, for hyperplanes H_i and H_j in $\mathcal{A}$, we have $H_i < H_j$ if and only if $i < j$. Let $I(\mathcal{A})$ be the ideal of $E(\mathcal{A})$ as defined previously, and let $A(\mathcal{A}):=E(\mathcal{A})/I(\mathcal{A})$ be the Orlik-Solomon algebra. Let $\pi: E(\mathcal{A})\to A(\mathcal{A})$ be the canonical projection. We write a_S to represent the image of e_S under π.

We demonstrate that $A(\mathcal{A})$ is a free graded K-module by defining the broken circuit basis for $A(\mathcal{A})$. By Theorem 1.9 to follow, this is indeed a basis for $A(\mathcal{A})$.

Definition 1.8. Let $S=\{H_{i_1},...,H_{i_p}\}$ be an ordered subset of A with $i_1 < \cdots < i_p$. We say a_S is basic in $A_p(\mathcal{A})$ if

1. S is independent, and
2. For any $1\leq k\leq p$, there does not exist a hyperplane $H\in\mathcal{A}$ so that $H<H_{i_k}$ with $\{H,H_{i_k},H_{i_{k+1}},...,H_{i_p}\}$ dependent.

The set of $\{a_S\}$ with S as above form the broken circuit basis for $A(\mathcal{A})$, whose name is justified by the following theorem.

Theorem 1.9. As a K-module, $A(\mathcal{A})$ is a free, graded module. The broken circuit basis forms a basis for $A(\mathcal{A})$.

Proof. This is proven in Theorem 3.55 in [3].

Example 1.10. Let dim $V = \ell$, and let $\mathcal{A}$ be the braid arrangement in V given by

$$Q(\mathcal{A}) = \prod_{1 \le i < j \le \ell} (x_i - x_j).$$

Let H_{ij} correspond to the hyperplane given by x_i-x_j = 0. Order the hyperplanes lexicographically; that is, $H_{ij} < H_{mn}$ if either i < m or i = m and j < n. We will write $a_{H_{ij}} = a_{ij}$ in $A_1(\mathcal{A})$.

In order to compute dim $A_p(\mathcal{A})$, we need to describe the elements of the broken circuit basis in $A_p(\mathcal{A})$. Let $a = a_{i_1 j_1} a_{i_2 j_p} \ldots a_{i_p j_p}$ be an element of the broken circuit basis in $A_p(\mathcal{A})$. By definition of the hyperplanes, we have $i_k < j_k$.

We first verify all the second indices of a are distinct. Suppose j1 = j2. Without loss of generality, we may assume $i_1 < i_2$. Then $\{H_{i_1 j_1}, H_{i_2}, _{j_2}, H_{j_1}, _{j_2}\}$ is dependent with $H_{i_1 i_2}$ being minimal in the set; this contradicts the assumption α is in the broken circuit basis. In a similar fashion, we have and will assume $j_1 < j_2 < \ldots < j_p$.

We now verify the first indices have no restriction other than $i_k<j_k$. Suppose $i_1 = i_2$, then $\{H_{i_1 j_1}, H_{i_2}, _{j_2}, H_{j_1}, _{j_2}\}$ is dependent; but the minimal element of this set is $H_{i_1 j_1}$. Notice $H_{i_1 j_1}, H_{i_2}, _{j_2}, H_{j_1}, _{j_2}$ is not basic as there are two of the second indices equal and this situation was eliminated. Therefore, a is still an element of the broken circuit basis as it does not contain the factor $a_{j_1 j_2}$. Hence, there are no restrictions on i_k other than $j_k > i_k$.

It is now just a matter of counting the possibilities we have for $\{i_1 j_1, \ldots, i_p j_p\}$ with the restrictions $j_1 < j_2 < \ldots < j_p$ and $i_k < j_k$ for k = 1, ..., p.

Fix $j_1, \ldots j_p$. There are $\ell - j_k$ choices for i_k for each k = 1, ..., p. Thus,

$$\dim A_p(\mathcal{A}) = \sum_{i_p = 1 + i_p - 1}^{\ell - 1} \ldots \sum_{i_2 = 1 + i_1}^{\ell - p + 1} \sum_{i_1 = 1}^{\ell - p} (\prod_{k=1}^{p} (\ell - j_k))$$
$$= \sum_{1 \le j_1 < j_2 < \ldots < j_p \le \ell - 1} j_1 j_2 .. j_p.$$

As usual, if $p = 0$, then this sum is taken to be 1.

The dimension s of $A_1(\mathcal{A})$ and $A_2(\mathcal{A})$ can be easily simplified. Obviously, we have dim $A_1(\mathcal{A}) = \binom{\ell}{2}$. For the dimension of $A_2(\mathcal{A})$,consider minimally dependent sets of three hyperplanes. Any such set must be of the form $\{H_{ij}, H_{ik}, H_{jk} : i < j < k\}$. There are $\binom{\ell}{3}$ of these sets. Hence, $A_2(\mathcal{A})$ = dim E_2-$\binom{\ell}{3}$. Using the fact, $n = \binom{\ell}{2}$ we arrive at dim $A_2(\mathcal{A}) = \frac{\ell(\ell - 1)(\ell - 2)(3\ell - 1)}{24}$.

Denition 1.11. Let $\mathcal{A}$ be an arrangement. Let $H_0 \in \mathcal{A}$. We define the arrangements given by deletion and restriction

$\mathcal{A}' = \{H : H \in \mathcal{A} \setminus H_0\}$ and

$\mathcal{A}'' = \{H_0 \cap H : H \in \mathcal{A}$ and $H \cap H_0 \neq \varnothing\}$.

Denition 1.12. Let $\pi(A(\mathcal{A}), t)$ be the Poincarè polynomial of the free graded K-module $A(\mathcal{A})$; that is, $\pi(A(\mathcal{A}), t) = \sum_{p=0}^{\ell} rank(A_p(\mathcal{A}))t^p$.

Theorem 1.13. Let $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ be a triple given by deletion and restriction. Then $\pi(A(\mathcal{A}), t) = \pi(A(\mathcal{A}'), t) + t\pi(A(\mathcal{A}''), t)$.

Proof. This is Corollary 3.67 in [3].

We end this section by furnishing two additional definitions which are needed in the subsequent section.

Definition 1.14. An element $X \in L(\mathcal{A})$ is said to be modular if for any $Y \in L(\mathcal{A})$ and any $Z \in L(\mathcal{A})$ with $Z \le Y$, we have
$Z \vee (X \wedge Y) = (Z \vee X) \wedge Y$.

Definition 1.15. Let $\mathcal{A}$ be an arrangement. We say $\mathcal{A}$ is supersolvable if $L(\mathcal{A})$ has a maximal chain of modular elements
$V = X_0 < X_1 < \ldots < X_\ell = \cap_{H \in \mathcal{A}} H$, while $rank(\mathcal{A}) = \ell$.

2. Main Theorem

Factorization of the Poincarè polynomial has been studied extensively. Stanley showed that supersolvable arrangements have Poincare polynomials that factor into linear factors [4]. A generalization of supersolvable arrangements gave a factorization into linear factors by looking at nice partitions [5]. Other generalizations of supersolvable arrangements are given in [1] and [2]. In this section, we show a factorization of the Poincarè polynomial when the arrangement has a special subarrangement which implies the existence of a modular element in $L(\mathcal{A})$.

Definition 2.1. Consider the following conditions on a nonempty subset $\mathcal{H} \subseteq \mathcal{A}$:

(A) for any $\{H_{i_1}, H_{i_2}\} \subseteq H$, there exists a unique $K \in \mathcal{A}$ with $K \notin \mathcal{H}$ and K containing $H_{i_1} \cap H_{i_2}$ and

(B) For any $\{K_{q_1}, \ldots, K_{q_m}\} \subseteq \mathcal{A} \setminus \mathcal{H}$, we have $\bigcap_{k=1}^{m} K_{qk}$ is contained in no hyperplanes from $\mathcal{H}$.

If such $\mathcal{H}$ exists in $\mathcal{A}$, we say $\mathcal{A}$ has the trio separation property under $\mathcal{H}$.

In the above definition, $Z = \cap_{H \in \mathcal{H}} H$ is a modular element of $L(\mathcal{A})$. See Stanley [4]. However, let $\mathcal{A}$ be the arrangement given by the hyperplanes {x, y, z, x + y − z} with $\mathcal{H}$ given by {z}. Then the hyperplane given by {z = 0} is a modular element but does not satisfy condition (B). Hence, modularity of $Z = \cap_{H \in \mathcal{H}} H$ does not imply that conditions (A) and (B) are satisfied.

Theorem 2.2. Suppose $\mathcal{H} \subset \mathcal{A}$ satisfies condition (A). There exists an ordering of the hyperplanes so that the broken circuit basis contains no elements $a_{\vec{v}}$ where $\vec{v}$ contains two indices corresponding to hyperplanes in $\mathcal{H}$.

Proof. Order the hyperplanes so that for any $H_i \in \mathcal{H}$ and any $H_k \in \mathcal{A} \setminus \mathcal{H}$, we have i > k. Let $a_{\vec{v}}$ be a basic element of $A(\mathcal{A})$ and suppose $\vec{v}$ contains two indices corresponding to hyperplanes in $\mathcal{H}$, say H_α and H_β. Since $\mathcal{H}$ satises condition (A), there exists $H_y \in \mathcal{A} \setminus \mathcal{H}$ with $H_\alpha \cap H_\beta \subset H_\gamma$. By our choice of ordering, $\gamma < \alpha, \beta$ and hence $a_{\vec{v}}$ is not basic.

Suppose $\mathcal{H} \subseteq \mathcal{A}$ satisfies condition (A). Let $X \in L(\mathcal{A})$ have rank greater than or equal to 2. Since $\mathcal{H}$ satisfies condition (A), we must have some hyperplanes containing X that are in $\mathcal{A} \setminus \mathcal{H}$. Let $X' \in L(\mathcal{A})$ represent the intersection of the hyperplanes containing X that are in $\mathcal{A} \setminus \mathcal{H}$.

Lemma 2.3. Supposes $\mathcal{A}$ has the trio separation property under $\mathcal{H}$. Let $X \in L(\mathcal{A})$ have rank greater than or equal to 2. Fix $H_0 \in \mathcal{H}$. Then $X' = (X \cap H_0)'$. Moreover, if $(X \cap H_0)' = (Y \cap H_0)'$ for any $X, Y \in L(\mathcal{A}) \setminus \{V, H_0\}$, then $X \cap H_0 = Y \cap H_0$.

Proof. Let $X \in L(\mathcal{A})$ have rank greater than or equal to 2. It is obvious that $X' \subseteq (X \cap H_0)'$. Suppose there is a hyperplane $H \in \mathcal{H}$ containing X. By condition (A), $(X \cap H_0)$ is a hyperplane and X' must contain at least one hyperplane, so $(X \cap H_0) = X'$. Suppose all hyperplanes containing X are in $\mathcal{A} \setminus \mathcal{H}$. Then $(X \cap H_0)$ is precisely X' by condition (B).

Suppose $(X \cap H_0)' = (Y \cap H_0)'$ for some $X, Y \in L(\mathcal{A}) \setminus \{V, H_0\}$. Suppose there exists $H \in \mathcal{H} \setminus \{H_0\}$ with H containing $X \cap H_0$. They by (A), $(H \cap H_0)'$ is a hyperplane containing $H \cap H_0$; hence, H contains $(Y \cap H_0)' \cap H_0$ which contains $Y \cap H_0$.

Lemma 2.4. Supposes $\mathcal{A}$ has the trio separation property under $\mathcal{H}$. Fix $H_0 \in \mathcal{H}$. Then $L(\mathcal{A}^{H0}) \cong L(\mathcal{A} \setminus \mathcal{H})$.

Proof. Let $\Phi : L(\mathcal{A}^{H0}) \to L(\mathcal{A} \setminus \mathcal{H})$ via $\Phi(X \cap H_0) = (X \cap H_0)'$ and $\Phi(H_0) = V$. To verify Φ is injective, suppose $(X \cap H_0)' = (Y \cap H_0)'$ for some $X, Y \in L(\mathcal{A}) \setminus \{V, H_0\}$. By Lemma 2.3, $X \cap H_0 = Y \cap H_0$.

To verify Φ is surjective, suppose $X \in L(\mathcal{A} \setminus \mathcal{H})$. Then $\Phi(X \cap H_0) = (X \cap H_0)' = X' = X$.

Furthermore, it is obvious that Φ is order preserving on the lattices. We are now ready to state and prove the following:

Theorem 2.5. Suppose $\mathcal{A}$ has the trio separation property under $\mathcal{H}$. The Poincare polynomial of $\mathcal{A}$ is computed via

$$\pi(A(\mathcal{A}), t) = (1 + |\mathcal{H}| \cdot t)\pi(A(\mathcal{A} \setminus \mathcal{H}), t).$$

Proof. We begin by applying Theorem 1.13 repeatedly to $\mathcal{H}$ = $\{H_1, \ldots, H_m\}$. It follows that

$$\pi(A(\mathcal{A}), t) = \pi(A(\mathcal{A} \setminus \mathcal{H}), t) + \sum_{i=1}^{m} t\pi(A(\mathcal{A} \setminus \{H_1, \ldots, H_{i-1}\})^{H_i}), t).$$

By Lemma 2.4,

$$\begin{aligned}\pi(A(\mathcal{A}), t) &= \pi(A(\mathcal{A} \setminus \mathcal{H}), t) + mt\pi(A(\mathcal{A} \setminus \mathcal{H}), t) \\ &= (1 + |\mathcal{H}| \cdot t)\pi(A(\mathcal{A} \setminus \mathcal{H}), t)\cdot\end{aligned}$$

Hence, we have computed the Poincarè polynomial of $A(\mathcal{A})$ in terms of the Poincarè polynomial of $A(\mathcal{A} \setminus \mathcal{H})$.

3. Examples

Denition 3.1. Let $\mathcal{A}_\ell$ be the braid arrangement dened by

$$Q(\mathcal{A}_\ell) = \prod_{1 \le i < j \le \ell} (x_i - x_j).$$

Lemma 3.2. Let $\mathcal{A}_\ell$ denote the braid arrangement. Let $H_{i,j}$ be the hyperplane determined by $x_i - x_j$ for $1 \le i < j \le \ell$. Then for any $2 \le \beta \le \ell$, we have:

$$(\mathcal{A}_\ell \setminus \{H_{1,\ell}, \ldots, H_{\beta,\ell}\})^{H_{\beta+1,\ell}} \cong \mathcal{A}_{\ell-1}.$$

Proof. Let $\mathcal{H} = \{H_{1,\ell}, \ldots, H_{\beta,\ell}\}$. Then $\mathcal{H}$ satises conditions (A) and (B). By Lemma 2.4, the result is immediate.

Theorem 3.3. Let A_ℓ denote the braid arrangement. Then

$$\pi(\mathcal{A}_\ell) = (1 + (\ell - 1) \cdot t)\pi(\mathcal{A}_{\ell-1}).$$

Proof. Let $\mathcal{H} = \{H_{1,\ell}, \ldots, H_{\beta,\ell+1}\}$. Then $\mathcal{A}$ has the trio separation property under $\mathcal{H}$. By Theorem 2.5 and Lemma 3.2, the result is immediate.

Definition 3.4. Let V be an $\ell-$ dimensional vector space over the finite field of q elements, $\mathbb{F}_q$. Let $\mathcal{A}_\ell$ be the central arrangement of all hyperplanes through the origin.

Lemma 3.5. Let $\mathcal{A}_\ell$ denote the arrangement defined in Definition 3.4. Let $\vec{c} = \{c_1, \ldots, c_{\ell-1}\}$ for $c_i \in \mathbb{F}_q$. Denote $H_{\vec{c},\ell}$ by the hyperplane determined by $x_\ell + \sum_{1 \le i \le \ell-1} c_i x_i$. Let $\mathcal{H}$ be the collection of hyperplanes $H_{\vec{c},\ell}$. For any $U \subset \mathcal{H}$ with $H_{\vec{c},\ell} \notin U$, we have

$$(\mathcal{A}_\ell \setminus U)^{H_{1,\vec{c}}} \cong \mathcal{A}_{\ell-1}.$$

Proof. Since $\mathcal{A}$ has the trio separation property under $\mathcal{H}$, the result is immediate by Lemma 2.4.

Theorem 3.6. Let $\mathcal{A}_\ell$ denote the arrangement of Denition 3.4. Then

$$\pi(\mathcal{A}_\ell) = (1 + q^{\ell-1}.t)\pi(\mathcal{A}_{\ell-1})\cdot$$

Proof. Take $\mathcal{H}$ to be the collection of hyperplanes $H_{\vec{c},\ell}$ as dened in Lemma 3.5. By Theorem 2.5 and Lemma 3.5, the result is immediate.

Competing interest: The authors declare that they have no competing interests.

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