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Contributions to Pure and Applied Mathematics

Mukai Conjecture

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In algebraic geometry, syzygies roughly refers to the relations of defining equations of a projective variety embedded in a projective space. They are defined in terms of a minimal graded free resolution for the homogeneous coordinate ring over a polynomial ring. Since the resolution is unique up to isomorphism, it makes sense to talk about the number of a minimal set of generators of given degree in any syzygy module.

The study of syzygies originated from D. Hilbert, and was developed by D. Mumford and his school, who introduced the notions of normally generated and normally presented. Later in 1980s, M. Green [1] introduced Koszul cohomology, putting the study in a more systematic way. In particular, Green's defined Property N_p, incorporating the notions of normally generated and normally presented as Properties N_0 and N_1 respectively. There has been a great deal of interest in the study of syzygies and its connection to Hodge theory, moduli of curves, and Hilbert schemes. The themes largely center around the syzygies for algebraic curves; famous conjectures include the Green's canonical curve conjecture, Green-Lazarfeld' secant conjecture, and Green-Lazarsfeld's gonality conjecture (see [2] for an excellent exposition). The gonality conjecture was completely solved by Ein-Lazarsfeld [3] after Voisin's seminar work. For varieties of arbitrary dimension, a conjecture, commonly attributed to S. Mukai, relates the positivity of adjoint linear series to the linearity of the minimal resolution.

To be more specific, we work over the field of complex numbers. Let X be a smooth projective variety and L a base point free line bundle. Any global section of L can be locally regarded as a holomorphic function. Suppose L is very ample line bundle on X, by definition, a basis $s_0, s_1, \cdots s_N$ for the space of global sections of L will give rise to an embedding from X to P^N. The embedding is called complete (If a proper subset of a basis yields an embedding, then the embedding is called incomplete). Let I_X denote the defining (saturated) ideal in the polynomial ring $S=\mathbb{C}[x_0, x_1, \cdots, x_N]$ for the image of X. We have the minimal graded free resolution of S-modules:

$$\cdots \to E_2 \to E_1 \to E_0 \to M \to 0,$$

Where $M = S/I_X$, $E_i = \otimes S(-a_{ij})$, a_{ij} are non-negative integers, recording the degrees of the minimum generators in the i-th syzygy module. Then Property N_p means that this resolution is simple up to step p. More precisely, it means that M coincides with the section ring for L, $E_0 = S$, defining equations are given by quadratic equations, that is, $E_1 = \otimes S(-2)$, and that all syzygies are linear, that is $E_i = \otimes S(-i-1)$ for 1≤i≤p. Green [Gre] proved:

Theorem: For any smooth curve of genus g, if the degree of line bundle L is at least 2g+1+p, then the completely linear system $|L|$ satisfies Property N_p.

The theorem unifies a number of classical results. For higher-dimensional varieties, we have

Mukai conjecture: For any n-dimensional smooth projective variety X and any ample divisor A, if the integer r is at least n+2+p, then the complete linear system of the adjoint divisor $K_X + rA$ satisfies Property N_p.

As extensions of the Fujita conjecture in syzygies study, the Mukai conjecture and its variants have generated a large amount of research over the decades (see [4, 5, 6, 7, 8, 9] and reference therein). Some notable advances in this direction include: Ein-Lazarsfeld [3] proved that the conjecture is true for arbitrary dimension provided A is very ample. Recently, Lacini-Purnaprajna [9] claimed a proof for the case when A is ample and base point free.

In dimension two, although the Fujita conjecture has been completely solved [10], the Mukai conjecture is still widely open. For some special surfaces, such as anti-canonical rational surfaces, K3, Abel surfaces, and some toric surfaces, the conjecture has positive and even finer answers. It is worth mentioning that the K3 situation was only recently solved by Agostini-Kuronya-Lozovanu [8]. However, there is so far no unified approach to handle all surfaces; and for surfaces of general type, there are only a few partial results.

In higher dimensions, most results are only for Abelian varieties. Pareschi [7] first proved the Lazarsfeld conjecture using Fourier-Mukai transformation, and then Pareschi-Popa [11, 12] developed the concept of M-regularity and extend Pareschi's results to a large extent. In recent years, based on the methods for higher-dimensional Fujita conjecture and the concept of cohomological rank function (cf. [13]), Ito [14], Caucci [15] and Jiang Zhi [16] established Reider type syzygies result on Abelian varieties.

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