Contributions to Pure and Applied Mathematics

Open Problems for Inscribed Cones and Circumscribed Cylinders of Centered Convex Bodies

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In this short note, I list some of the open problems (minmax and maxmin inequalities) for inscribed cones and circumscribed cylinders of centered convex bodies. I also state what is known about them, and how they can be used (if possible) to solve other open problems in convex geometry. The following notations will be used throughout the note, and it will be assumed that $d \geq 3$. A convex set $K \subset \mathbb{R}^d$ is a convex body if its interior is nonempty. A convex body B is called centered if $-B = B$ (i.e., it is symmetric with respect to the origin). The symbol $\lambda_i(\cdot)$ will be used for the i-dimensional Lebesgue measure (volume) in $\mathbb{R}^d$, where $1 \leq i \leq d$, and when $i = d$ the subscript will be omitted. For a given direction $u \in S^{d-1}$, $u^{\perp}$ will be used for the $(d-1)$-dimensional hyperplane (passing through the origin) orthogonal to u, and by l_u the 1-subspace parallel to u. Furthermore, $\lambda_1(K \mid l_u)$ denotes the width of K at u, and $\lambda_{d-1}(B \mid u^{\perp})$ the $(d-1)$-dimensional outer cross-section measure or brightness of B at $u \in S^{d-1}$, where $B \mid u^{\perp}$ is the orthogonal projection of B onto $u^{\perp}$ (more about these notations the reader should refer to [2]). The space $\mathbb{R}^d$ and its dual space $\mathbb{R}^{d*}$ will be identified by using the standard basis. In that case, $\lambda_i(\cdot)$ and $\lambda_i^*(\cdot)$ coincide in $\mathbb{R}^d$. The i-dimensional volume of the unit ball in $\mathbb{R}^i$ will be denoted by the symbol $\in_i$.

Let $C(B,u)$ be the unbounded cylinder circumscribed about B (i.e., the union of all lines parallel to u and intersecting B) generated by u. We define $B(u)$ to be a compact circumscribed cylinder of B obtained from $C(B,u)$ which is bounded by the two parallel supporting hyperplanes of B at the intersection of l_u (i.e., 1-subspace parallel to u)with the boundary of B. We can select these two parallel supporting hyperplanes of B perpendicular to lu if hyperplanes are not uniquely defined. For a given unit vector u, one can also construct the inscribed cone of maximal volume with base $B \cap u^{\perp}$ and apex in B. The apex of such a cone is a point of B on a supporting hyperplane parallel to $u^{\perp}$.

Petty [6] proved the following maxmin inequality for circumscribed cylinders of a centered convex body B:

$$\max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \mid u^{\perp})\lambda_1(B \mid l_u)}{\lambda(B)} \geq \frac{2 \in_{d-1}}{\in_d},$$

with equality if and only if K is an ellipsoid (see also [7], [4] for all convex bodies).

Lutwak [3] (see also [5]) proved the following minimax inequality for double-cones inscribed in a centered convex body B of $\mathbb{R}^d$:

$$\min_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^{\perp})\lambda_1(B \mid l_u)\lambda(B^{\circ}) \leq 2 \in_d \in_{d-1},$$

with equality if and only if B is an ellipsoid. Here B° stands for the polar body of B.

Problem 1. What is the minimum value of

$$\max_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^{\perp})\lambda_1(B \mid l_u)\lambda(B)^{-1} ?$$

It is reasonable to conjecture that the minimum is $2 \in d - 1 \in_d^{-1}$ ([1]).

Problem 2. What is the maximum value of

$$\min_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^{\perp})\lambda_1(B \mid l_u)\lambda(B)^{-1} ?$$

It is also reasonable to conjecture that the maximum is attained for centered ellipsoids (see [1]).

Problem 3. What is the maximum value of

$$\min_{u \in S^{d-1}} \lambda_{d-1}(B \mid u^{\perp})\lambda_1(B \cap l_u)\lambda(B)^{-1} ?$$

Problem 4. What is the minimum value of

$$\max_{u \in S^{d-1}} \lambda_{d-1}(B \mid u^{\perp})\lambda_1(B \cap l_u)\lambda(B^{\circ}) ?$$

Problem 5. What is the maximum value of

$$\min_{u \in S^{d-1}} \lambda_{d-1}(B \mid u^{\perp})\lambda_1(B \cap l_u)\lambda(B^{\circ}) ?$$

Problem 6. What is the minimum value of

$$\max_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^{\perp})\lambda_1(B \mid l_u)\lambda(B^{\circ}) ?$$

One of the challenging problems in convex geometry is that whether a centered convex body B and ΠB° (i.e., projection body of B°) are homothetic if and only if B is an ellipsoid. If one can prove that

$$\max_{u\in S^{d-1}} \lambda_{d-1}(B\cap u^\perp)\lambda_1(B\,|\,l_u)\lambda(B^\circ)\geq 2\in_{d-1}\in_d,$$

with equality if and only if B is an ellipsoid, then B and ΠB° are homothetic if and only if B is an ellipsoid.

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