



Contributions to Pure and Applied Mathematics

Unique Continuation for the Multi-term Time Fractional Diffusion Equation and its Numerical Simulation

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Abstract

This paper deals with the unique continuation principle of solutions for a one-dimensional anomalous diffusion equation with multi-term time fractional derivatives. The proof is mainly based on the Laplace transform and Theta function method. Numerically, we reformulate the unique continuation as an optimization problem, and propose an iterative thresholding algorithm to simulate it numerically. Finally, several numerical experiments are presented to show the accuracy and efficiency of the algorithm.

keywords: fractional diffusion equation, unique continuation principle, fractional Theta function, Laplace transform, Phragmén-Lindelöf principle

1 Introduction and main results

In this paper, letting $T > 0$, $\ell \in \mathbb{N}^+$ and α_j, q_j ($j = 1, 2, \dots, \ell$) be positive constants such that $0 < \alpha_\ell < \dots < \alpha_1 < 1$, we consider the following one-dimensional time-fractional diffusion equation with multi-term time fractional derivatives

$$\sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} u(x, t) - \partial_x^2 u(x, t) = 0, \quad (x, t) \in (0, 1) \times (0, T]. \quad (1.1)$$

Here by $\partial_t^{\alpha_j}$ we denote the Caputo derivative (see, e.g., Podlubny [17 §2.4.1])

$$\partial_t^{\alpha_j} g(t) := \frac{1}{\Gamma(1 - \alpha_j)} \int_0^t \frac{g'(\tau)}{(t - \tau)^{\alpha_j}} d\tau, \quad t > 0,$$

where $\Gamma(\cdot)$ stands for the Gamma function.

Unlike the usual parabolic equations characterized by exponential decay in time and Gaussian profile in space, fractional diffusion equations possess properties of slow decay in time and long-tailed profile in space. Therefore, fractional diffusion equations are more suitable model equations than parabolic equations for modeling anomalous diffusion with non-Fickian growth rates and long-tailed profiles (see e.g., Adams and Gelhar[1] and Hatano and Hatano [6]). Theoretical research has developed rapidly, and it has been proven that most properties of parabolic equations, such as well-posedness, (Strong) Maximum principle, analyticity, and asymptotic behavior, have been parallelly extended to the fractional order case. see, e.g., Liu, Rundell and Yamamoto [14], Luchko [15], Sakamoto and Yamamoto [19] for the single-term time fractional diffusion equations (that is $\ell = 1$). For the multi-term case, one can refer to Li, Liu and Yamamoto [9], Liu [12], Luchko [16] and the references therein.

In addition to the aforementioned aspects, the unique continuation principle (UCP) is also one of the remarkable characteristics of parabolic equations. In recent years, there has been increasing interest in extending this theory to fractional order equations, given their widespread application in various areas of science and engineering. For example The UCP was used to solve the uniqueness of some inverse problems on determining the spatially dependent component of the sources ([7, 13, 20] and the references there in). Not only for the inverse problems but also for the approximate control problems, one of the very important key in dealing with these problems is the UCP, see e.g., Fujishiro and Yamamoto [5].

To the best knowledge of the authors, the UCP for the multi-term case has not yet been established. With this in mind, we aim to generalize this property to the fractional case. Building on Laplace transforms and fractional Theta functions, we have

Theorem 1. *Let I be a nonempty open subinterval of $(0, 1)$ and $2\alpha_\ell > \alpha_1$. We suppose $u \in L^2(0, T; H^2(0, 1))$ is a solution to the fractional diffusion equation (1.1). Then $u = 0$ in $(0, 1) \times (0, T)$ provided that $u \equiv 0$ in $I \times [0, T]$.*

Uniqueness result similar to Theorem 1 was established in Li and Yamamoto [10], where the one-dimensional time-fractional diffusion equation was studied, and the uniqueness of solutions was shown using the Theta function method. For the general case of multiple dimensions, there is currently no positive answer regarding the UCP. However, under some additional assumptions on the solutions, one can still obtain similar uniqueness results. We refer to Jiang, Li, Liu and Yamamoto [7] and Sakamoto and Yamamoto [19] in which the homogeneous Dirichlet boundary condition on the entire boundary was required and the existence of eigensystem provides convenience for the argument. On the other hand, in the case of the homogeneous initial condition, the UCP was proved in Cheng, Lin and Nakamura [3] for half-order fractional diffusion equation using Carleman estimates for the

operator $\partial_t - \Delta^2$. For a general fractional order in the $(0, 1)$ interval, Lin and Nakamura [11] recently established the UCP by applying a newly developed Carleman estimate based on calculus of pseudo-differential operators.

The remainder of this paper is organized as follows. In Section 2, we introduce all necessary background information about the fundamental solution of (1.1) and the fractional Theta function. Then, Section 3 is devoted to the proof of the main theorem. In Section 4.1, we propose an iterative thresholding algorithm for numerically simulating the UCP of the solution. This is followed by several numerical examples in Section 4.2 that illustrate the performance of the proposed method. Finally, a concluding remark is given in Section 5.

2 Preliminary material

2.1 Fundamental solution

Let $K(x, t)$ be the fundamental solution of the following free space-time fractional diffusion equation

$$\begin{cases} \sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} K(x, t) = \partial_x^2 K(x, t), & x \in \mathbb{R}, t > 0, \\ K(x, 0) = \delta(x), & x \in \mathbb{R}, \end{cases} \quad (2.1)$$

where we denote $\delta(x)$ as the Dirac delta function. We consider the Laplace transform of the fundamental solution $K(x, t)$ with respect to time variable t . For this, we apply Laplace transform on both sides of the equation (2.1) and then

$$\begin{cases} \sum_{j=1}^{\ell} q_j s^{\alpha_j} \widehat{K}(s) = \partial_x^2 \widehat{K}(s) + \sum_{j=1}^{\ell} q_j s^{\alpha_j - 1} \delta(x), \\ K(x, 0) = \delta(x), & x \in \mathbb{R}. \end{cases}$$

Then taking Fourier transform \mathcal{F} , and noting that the formula $\mathcal{F}[\delta] = 1$ and $\mathcal{F}[\partial_x^2 K](\xi) = -\xi^2 \mathcal{F}[K](\xi)$, we see that

$$\sum_{j=1}^{\ell} q_j s^{\alpha_j} \mathcal{F}[\widehat{K}(s)] = -\xi^2 \mathcal{F}[\widehat{K}(s)] + \sum_{j=1}^{\ell} q_j s^{\alpha_j - 1}.$$

We denote $\phi(s) := \sum_{j=1}^{\ell} q_j s^{\alpha_j}$, then we can rephrase the above them as follows

$$\phi(s) \mathcal{F}[\widehat{K}(s)] = -\xi^2 \mathcal{F}[\widehat{K}(s)] + s^{-1} \phi(s).$$

Therefore

$$\mathcal{F}[\widehat{K}(s)](\xi) = \frac{s^{-1} \phi(s)}{\phi(s) + \xi^2}.$$

Using the Fourier inverse transform, we obtain the Laplace transform \widehat{K} of the fundamental solution to (2.1)

$$\widehat{K}(s) = \mathcal{F}^{-1} \left[\frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} d\xi.$$

We next calculate the complex integral $\mathcal{F}^{-1} \left[\frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} \right]$.

Lemma 2.1.

$$\mathcal{F}^{-1} \left[\frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} \right] (x) = \frac{1}{2s} \phi^{\frac{1}{2}}(s) \exp\{-\phi^{\frac{1}{2}}(s)|x|\}.$$

Proof. From the inverse Fourier transform formula, we have

$$\mathcal{F}^{-1} \left[\frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} \right] (x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} d\xi.$$

We denote the contour from $-R$ to R by C_0 , the semicircle with radius R in the upper and lower half plane by C_{R+} and C_{R-} respectively. Also let C_+, C_- be the closed contours which consist of C_0, C_{R+} and C_0, C_{R-} respectively.

For the case of $x > 0$, working on the closed contour C_+ , we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} d\xi &= \lim_{R \rightarrow +\infty} \frac{1}{2\pi} \oint_{C_+} e^{ix\xi} \frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} d\xi - \lim_{R \rightarrow +\infty} \frac{1}{2\pi} \int_{C_{R+}} e^{ix\xi} \frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} d\xi \\ &= \lim_{R \rightarrow +\infty} \frac{1}{2\pi} \oint_{C_+} e^{ix\xi} \frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} d\xi, \end{aligned}$$

where the second limit is 0 as follows from Jordan's Lemma. Since $0 < \alpha_j < 1$ and $q_j > 0$, by our assumption, we have $\Re\phi(s) = \sum_{j=1}^{\ell} q_j |s|^{\alpha_j} \cos \alpha_j \theta \geq 0$, which in turns leads to $\Re\phi^{\frac{1}{2}}(s) \geq 0$.

Then there is only one singular point $\xi = i\phi^{\frac{1}{2}}(s)$ in C_+ which is continued by the upper half plane. By the residue theorem, we have

$$\begin{aligned} \lim_{R \rightarrow +\infty} \frac{1}{2\pi} \oint_{C_+} e^{ix\xi} \frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} d\xi &= \lim_{R \rightarrow +\infty} 2\pi i \frac{1}{2\pi} \exp \left\{ ix(i\phi^{\frac{1}{2}}(s)) \right\} \frac{s^{-1}\phi(s)}{2i\phi^{\frac{1}{2}}(s)} \\ &= \frac{1}{2s} \phi^{\frac{1}{2}}(s) \exp \left\{ -\phi^{\frac{1}{2}}(s)x \right\}. \end{aligned}$$

For the case of $x < 0$, similarly, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} d\xi = \frac{1}{2s} \phi^{\frac{1}{2}}(s) \exp \left\{ \phi^{\frac{1}{2}}(s)x \right\}.$$

Therefore,

$$\mathcal{F}^{-1} \left[\frac{s^{-1}\phi(s)}{\phi(s) + \xi^2} \right] (x) = \frac{1}{2s} \phi^{\frac{1}{2}}(s) \exp \left\{ -\phi^{\frac{1}{2}}(s)|x| \right\},$$

which completes the proof of the lemma. \square

From the above lemma, it follows that the fundamental solution $K(x, t)$ has the following formula of the Laplace transform with respect to t ,

$$\widehat{K}(x, s) = \frac{\phi^{\frac{1}{2}}(s)}{2s} \exp \left\{ -\phi^{\frac{1}{2}}(s)|x| \right\}, \quad x \in \mathbb{R}, \quad \phi(s) = \sum_{j=1}^{\ell} q_j s^{\alpha_j}.$$

Lemma 2.2. *The functions $K(x, t)$ is even with respect to x and satisfies the following estimates*

$$|K(x, t)| \leq C \sum_{j=1}^{\ell} t^{-\frac{\alpha_j}{2}}, \quad (x, t) \in \mathbb{R} \times (0, \infty), \quad (2.2)$$

and for the Riemann-Liouville derivative $D_t^{1-\alpha} K(x, t)$ with $\alpha > \frac{\alpha_1}{2}$:

$$|D_t^{1-\alpha} K(x, t)| \leq C \sum_{j=1}^{\ell} t^{\alpha - \frac{\alpha_j}{2} - 1}, \quad (x, t) \in \mathbb{R} \times (0, \infty), \quad (2.3)$$

where the constant $C > 0$ is independent of t , but may depend on $\alpha, \{\alpha_j\}_{j=1}^{\ell}$.

Proof. From the Laplace transform of $K(x, t)$:

$$\widehat{K}(x, t) = \frac{1}{2} s^{-1} \phi^{\frac{1}{2}}(s) \exp \left\{ -\phi^{\frac{1}{2}}(s)|x| \right\},$$

we conclude from the Fourier-Millin formula (e.g., Schiff [22]) for the inversion Laplace transform that $K(x, t) = \mathcal{L}^{-1}[\widehat{K}(s)]$ admits

$$K(x, t) = \frac{1}{4\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} s^{-1} \phi^{\frac{1}{2}}(s) \exp \left\{ -\phi^{\frac{1}{2}}(s)|x| \right\} e^{st} ds,$$

where $\gamma > 0$ is any fixed constant, from which we can directly derive the following estimate

$$|K(x, t)| \leq \frac{1}{4\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} |s|^{-1} |\phi^{\frac{1}{2}}(s)| \exp \left\{ -\frac{\sqrt{2}}{2} |\phi^{\frac{1}{2}}(s)||x| \right\} |ds| e^{\gamma t}.$$

Moreover, from the definition of $\phi(s)$, we see that

$$|\phi(s)| \geq \sum_{j=1}^{\ell} q_j |s|^{\alpha_j} \cos \alpha_j \theta \geq \sum_{j=1}^{\ell} q_j |s|^{\alpha_j} \cos \alpha_j \frac{\pi}{2} \geq c_0 \sum_{j=1}^{\ell} |s|^{\alpha_j},$$

where $c_0 := \min_{1 \leq j \leq \ell} \{q_j \cos \frac{\alpha_j}{2} \pi\}$. Therefore,

$$\begin{aligned} |K(x, t)| &\leq \frac{1}{4\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} |s|^{-1} \left(\sum_{j=1}^{\ell} q_j |s|^{\alpha_j} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\sqrt{2}}{2} c_0^{\frac{1}{2}} \left(\sum_{j=1}^{\ell} |s|^{\alpha_j} \right)^{\frac{1}{2}} |x| \right\} |ds| e^{\gamma t} \\ &\leq \frac{(\max\{q_j\})^{\frac{1}{2}}}{4\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} |s|^{-1} \left(\sum_{j=1}^{\ell} |s|^{\alpha_j} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\sqrt{2}}{2} |x| c_0^{\frac{1}{2}} \left(\sum_{j=1}^{\ell} |s|^{\alpha_j} \right)^{\frac{1}{2}} \right\} |ds| e^{\gamma t}. \end{aligned}$$

For short, we denote $c_1 := \frac{\sqrt{2}}{2}c_0^{\frac{1}{2}}$ and nothing that $s = \gamma + i\eta$, $\eta \in (-\infty, +\infty)$, we have

$$\begin{aligned} |K(x, t)| &\leq \frac{(\max\{q_j\})^{\frac{1}{2}}}{4\pi} \int_{-\infty}^{\infty} \frac{[\sum_{j=1}^{\ell}(\gamma^2 + \eta^2)^{\frac{\alpha_j}{2}}]^{\frac{1}{2}}}{(\gamma^2 + \eta^2)^{\frac{1}{2}}} \exp \left\{ -c_1|x| \left[\sum_{j=1}^{\ell}(\gamma^2 + \eta^2)^{\frac{\alpha_j}{2}} \right]^{\frac{1}{2}} \right\} d\eta e^{\gamma t} \\ &= \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \int_0^{\infty} \frac{[\sum_{j=1}^{\ell}(\gamma^2 + \eta^2)^{\frac{\alpha_j}{2}}]^{\frac{1}{2}}}{(\gamma^2 + \eta^2)^{\frac{1}{2}}} \exp \left\{ -c_1|x| \left[\sum_{j=1}^{\ell}(\gamma^2 + \eta^2)^{\frac{\alpha_j}{2}} \right]^{\frac{1}{2}} \right\} d\eta e^{\gamma t} \\ &= \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \int_0^{\infty} \left[\sum_{j=1}^{\ell}(\gamma^2 + \eta^2)^{\frac{\alpha_j}{2}-1} \right]^{\frac{1}{2}} \exp \left\{ -c_1|x| \left[\sum_{j=1}^{\ell}(\gamma^2 + \eta^2)^{\frac{\alpha_j}{2}} \right]^{\frac{1}{2}} \right\} d\eta e^{\gamma t}. \end{aligned}$$

By changing the variable η/γ to η , we find

$$|K(x, t)| \leq \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \int_0^{\infty} \left[\sum_{j=1}^{\ell}(1 + \eta^2)^{\frac{\alpha_j}{2}-1} \gamma^{\alpha_j} \right]^{\frac{1}{2}} \exp \left\{ -c_1|x| \left[\sum_{j=1}^{\ell}(1 + \eta^2)^{\frac{\alpha_j}{2}} \gamma^{\alpha_j} \right]^{\frac{1}{2}} \right\} d\eta e^{\gamma t}.$$

We divide the integral on the right-hand side of above inequalities into two parts:

$$\begin{aligned} &|K(x, t)| \\ &\leq \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \int_0^1 \left[\sum_{j=1}^{\ell}(1 + \eta^2)^{\frac{\alpha_j}{2}-1} \gamma^{\alpha_j-2} \right]^{\frac{1}{2}} \exp \left\{ -c_1|x| \left[\sum_{j=1}^{\ell}(1 + \eta^2)^{\frac{\alpha_j}{2}} \gamma^{\alpha_j} \right]^{\frac{1}{2}} \right\} d\eta e^{\gamma t} \gamma \\ &\quad + \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \int_1^{\infty} \left[\sum_{j=1}^{\ell}(1 + \eta^2)^{\frac{\alpha_j}{2}-1} \gamma^{\alpha_j-2} \right]^{\frac{1}{2}} \exp \left\{ -c_1|x| \left[\sum_{j=1}^{\ell}(1 + \eta^2)^{\frac{\alpha_j}{2}} \gamma^{\alpha_j} \right]^{\frac{1}{2}} \right\} d\eta e^{\gamma t} \gamma. \end{aligned}$$

We denote the two terms on the right hand side of the above inequality as I_1 and I_2 respectively. We will estimate each of them separately. Firstly, for I_1 , we have

$$\begin{aligned} I_1(x, t) &\leq \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \int_0^1 \left(\sum_{j=1}^{\ell} \gamma^{\alpha_j} \right)^{\frac{1}{2}} \exp \left\{ -c_1|x| \left(\sum_{j=1}^{\ell} \gamma^{\alpha_j} \right)^{\frac{1}{2}} \right\} d\eta e^{\gamma t} \\ &= \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \left(\sum_{j=1}^{\ell} \gamma^{\alpha_j} \right)^{\frac{1}{2}} \exp \left\{ -c_1|x| \left(\sum_{j=1}^{\ell} \gamma^{\alpha_j} \right)^{\frac{1}{2}} + \gamma t \right\}. \end{aligned}$$

If $\gamma t \leq 1$, a direct calculation implies

$$I_1(x, t) \leq \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \left(\sum_{j=1}^{\ell} \gamma^{\alpha_j} \right)^{\frac{1}{2}} e^{\gamma t}$$

$$= \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \sum_{j=1}^{\ell} (\gamma t)^{\frac{\alpha_j}{2}} t^{-\frac{\alpha_j}{2}} e^{\gamma t} \leq \frac{e \max\{q_j^{\frac{1}{2}}\}}{2\pi} \sum_{j=1}^{\ell} t^{-\frac{\alpha_j}{2}}, \quad t > 0.$$

If $\gamma t > 1$, we take $\gamma = \gamma_*$ s.t. $\gamma_* t = \frac{c_1}{2} |x| \left(\sum_{j=1}^{\ell} \gamma_*^{\alpha_j} \right)^{\frac{1}{2}}$ therefore we have

$$I_1(x, t) \leq \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \left(\sum_{j=1}^{\ell} \gamma_*^{\alpha_j} \right)^{\frac{1}{2}} e^{-\gamma_* t} = \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \left[\sum_{j=1}^{\ell} (\gamma_* t)^{\alpha_j} t^{-\alpha_j} \right]^{\frac{1}{2}} e^{-\gamma_* t}.$$

Now from the Cauchy-Schwartz inequality, and noting the inequality

$$\sum_{j=1}^{\ell} t^{\alpha_j} e^{-t} \leq \sum_{j=1}^{\ell} (\alpha_j)^{\alpha_j} e^{-\alpha_j} \leq \ell, \quad t > 0,$$

we see that

$$\begin{aligned} I_1(x, t) &\leq \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \left[\sum_{j=1}^{\ell} (\gamma_* t)^{\alpha_j} \right]^{\frac{1}{2}} \left(\sum_{j=1}^{\ell} t^{-\alpha_j} \right)^{\frac{1}{2}} e^{-\gamma_* t} \\ &\leq \frac{\ell \max\{q_j^{\frac{1}{2}}\}}{2\pi} \left(\sum_{j=1}^{\ell} t^{-\alpha_j} \right)^{\frac{1}{2}} \leq \frac{\ell \max\{q_j^{\frac{1}{2}}\}}{2\pi} \sum_{j=1}^{\ell} t^{-\frac{\alpha_j}{2}}, \quad t > 0. \end{aligned}$$

For I_2 , we see that

$$I_2(x, t) \leq \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \int_1^{\infty} \left[\sum_{j=1}^{\ell} \eta^{\alpha_j - 2} \gamma^{\alpha_j - 2} \right]^{\frac{1}{2}} \exp \left\{ -c_1 |x| \left[\sum_{j=1}^{\ell} (1 + \eta^{\alpha_j}) \gamma^{\alpha_j} \right]^{\frac{1}{2}} \right\} d\eta e^{\gamma t}.$$

Moreover, by a direct calculation, it is not difficult to see that

$$\left(\sum_{j=1}^{\ell} a_j \right)^{\frac{1}{2}} \leq \sum_{j=1}^{\ell} a_j^{\frac{1}{2}}, \quad \text{for } a_j \geq 0,$$

and then

$$\begin{aligned} &I_2(x, t) \\ &\leq \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \int_1^{\infty} \left(\sum_{j=1}^{\ell} \eta^{\frac{\alpha_j}{2} - 1} \gamma^{\frac{\alpha_j}{2}} \right) \exp \left\{ -c_1 |x| \sum_{j=1}^{\ell} (1 + \eta^{\frac{\alpha_j}{2}}) \gamma^{\frac{\alpha_j}{2}} \right\} d\eta e^{\gamma t} \\ &= \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi} \int_1^{\infty} \left(\sum_{j=1}^{\ell} \eta^{\frac{\alpha_j}{2} - 1} \gamma^{\frac{\alpha_j}{2}} \right) \exp \left\{ -c_1 |x| \sum_{j=1}^{\ell} \eta^{\frac{\alpha_j}{2}} \gamma^{\frac{\alpha_j}{2}} \right\} d\eta \exp \left\{ \gamma t - c_1 |x| \sum_{j=1}^{\ell} \gamma^{\frac{\alpha_j}{2}} \right\} \\ &\leq \frac{\max\{q_j^{\frac{1}{2}}\}}{\pi \min\{\alpha_j\}} \int_1^{\infty} \exp \left\{ -c_1 |x| \sum_{j=1}^{\ell} \eta^{\frac{\alpha_j}{2}} \gamma^{\frac{\alpha_j}{2}} \right\} d \left(\sum_{j=1}^{\ell} \eta^{\frac{\alpha_j}{2}} \gamma^{\frac{\alpha_j}{2}} \right) \exp \left\{ \gamma t - c_1 |x| \sum_{j=1}^{\ell} \gamma^{\frac{\alpha_j}{2}} \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\max\{q_j^{\frac{1}{2}}\}}{\pi \min\{\alpha_j\}} \frac{\exp\left\{-c_1|x|\sum_{j=1}^{\ell}\gamma^{\frac{\alpha_j}{2}}\right\}}{c_1|x|} \exp\left\{\gamma t - c_1|x|\sum_{j=1}^{\ell}\gamma^{\frac{\alpha_j}{2}}\right\} \\
&= \frac{\sqrt{2}\max\{q_j^{\frac{1}{2}}\}}{\pi\sqrt{c_0}\min\{\alpha_j\}|x|} \exp\left\{\gamma t - c_1|x|\sum_{j=1}^{\ell}\gamma^{\frac{\alpha_j}{2}}\right\}.
\end{aligned}$$

If $\gamma t > 1$, we take $\gamma = \gamma_*$ s.t. $\gamma_* t = \frac{c_1}{2}|x|\sum_{j=1}^{\ell}\gamma_*^{\frac{\alpha_j}{2}}$, therefore

$$\begin{aligned}
I_2(x, t) &\leq \frac{\sqrt{2}\max\{q_j^{\frac{1}{2}}\}}{\pi\sqrt{c_0}\min\{\alpha_j\}|x|} e^{-\gamma_* t} \leq \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi\min\{\alpha_j\}} \frac{1}{t} \sum_{j=1}^{\ell} \gamma_*^{\frac{\alpha_j}{2}-1} e^{-\gamma_* t} \\
&= \frac{\max\{q_j^{\frac{1}{2}}\}}{2\pi\min\{\alpha_j\}} t^{-1} \sum_{j=1}^{\ell} (\gamma_* t)^{\frac{\alpha_j}{2}-1} t^{1-\frac{\alpha_j}{2}} e^{-\gamma_* t} \leq C \sum_{j=1}^{\ell} t^{-\frac{\alpha_j}{2}}, \quad t > 0.
\end{aligned}$$

If $\gamma t \leq 1$, we take $\gamma = \gamma_*$ s.t. $\gamma_* t = 2c_1|x|\sum_{j=1}^{\ell}\gamma_*^{\frac{\alpha_j}{2}}$, therefore

$$\begin{aligned}
I_2(x, t) &\leq \frac{C}{|x|} \exp\left\{-c_1|x|\sum_{j=1}^{\ell}\gamma_*^{\frac{\alpha_j}{2}}\right\} \\
&= \frac{C}{|x|\sum_{j=1}^{\ell}\gamma_*^{\frac{\alpha_j}{2}}} \exp\left\{-c_1|x|\sum_{j=1}^{\ell}\gamma_*^{\frac{\alpha_j}{2}}\right\} \sum_{j=1}^{\ell} \gamma_*^{\frac{\alpha_j}{2}} \\
&\leq C \sum_{j=1}^{\ell} \gamma_*^{\frac{\alpha_j}{2}} = C \sum_{j=1}^{\ell} (\gamma_* t)^{\frac{\alpha_j}{2}} t^{-\frac{\alpha_j}{2}} \leq C \sum_{j=1}^{\ell} t^{-\frac{\alpha_j}{2}}, \quad t > 0,
\end{aligned}$$

which finishes the proof of the first part of the lemma.

For evaluating $D_t^{1-\alpha}K$ with $\alpha > \frac{\alpha_1}{2}$, we first calculate its Laplace transform $\mathcal{L}[D_t^{1-\alpha}K]$. Indeed, in view of the formula

$$\mathcal{L}\{D_t^\alpha f(t); s\} = s^\alpha \mathcal{L}\{f(t); s\} - J^{1-\alpha} f(0+).$$

Moreover, since $|K(x, t)| \leq C \sum_{j=1}^{\ell} t^{-\frac{\alpha_j}{2}}$, we have $J^\alpha K(x, t)$ tends to 0 as $t \rightarrow 0$, hence a direct calculation yields

$$\mathcal{L}\{D_t^{1-\alpha}K(x, t); s\} = s^\alpha \widehat{K}(s).$$

Now following the above argument for (2.2), the desired result in (2.3) can be proved. \square

Remark 2.1. *This result parallelly generalized the estimates of the fundamental solution in Kochubei [8, Proposition 1] to the multi-term time-fractional case. However, based on our calculation, we must assume a condition $2\alpha_\ell > \alpha_1$. The estimate for the fundamental solution without this assumption remains open.*

2.2 Fractional Theta function

A representation of the solution to the initial-boundary value problem in a bounded domain can be constructed via the usual Theta function for the heat equation Cannon [2]. We consider the generalization of this function by

$$\theta(x, t) = \sum_{m=-\infty}^{\infty} K(x + 2m, t), \quad t > 0.$$

Now investigate the Laplace transform of the fractional Theta function θ with respect to time variable $t \in (0, \infty)$. Indeed, from the definition of θ and the Laplace transform of the fundamental solution to (2.1), we see that

$$2\mathcal{L}\{\theta(x, t); s\} = s^{-1}\phi^{\frac{1}{2}}(s) \sum_{m=-\infty}^{\infty} e^{-|x+2m|\phi^{\frac{1}{2}}(s)}.$$

For $x \in [0, 1]$, we further have

$$\begin{aligned} 2\mathcal{L}\{\theta(x, t); s\} &= s^{-1}\phi^{\frac{1}{2}}(s) \left(e^{x\phi^{\frac{1}{2}}(s)} \sum_{m=-\infty}^{-1} e^{2m\phi^{\frac{1}{2}}(s)} + e^{-x\phi^{\frac{1}{2}}(s)} \sum_{m=0}^{\infty} e^{-2m\phi^{\frac{1}{2}}(s)} \right) \\ &= \frac{\phi^{\frac{1}{2}}(s)}{s} \left(e^{x\phi^{\frac{1}{2}}(s)} \sum_{m=1}^{\infty} e^{-2m\phi^{\frac{1}{2}}(s)} + e^{-x\phi^{\frac{1}{2}}(s)} \sum_{m=0}^{\infty} e^{-2m\phi^{\frac{1}{2}}(s)} \right) \\ &= \frac{\phi^{\frac{1}{2}}(s)}{s} \frac{e^{x\phi^{\frac{1}{2}}(s)}}{e^{2\phi^{\frac{1}{2}}(s)} - 1} + \frac{\phi^{\frac{1}{2}}(s)}{s} \frac{e^{(2-x)\phi^{\frac{1}{2}}(s)}}{e^{2\phi^{\frac{1}{2}}(s)} - 1}, \quad s > 0. \end{aligned}$$

From the definition of the fractional Theta function, it is not difficult to see that $\theta(1, 0) = 0$. Then using the formula

$$\mathcal{L}\{D_t^{1-\alpha_j}\theta(1, t); s\} = s^{1-\alpha_j}\mathcal{L}\{\theta(1, t); s\},$$

we are led to

$$\mathcal{L}\{D_t^{1-\alpha_j}\theta(1, t); s\} = \frac{\phi^{\frac{1}{2}}(s)}{s^{\alpha_j}} \frac{e^{\phi^{\frac{1}{2}}(s)}}{e^{2\phi^{\frac{1}{2}}(s)} - 1}, \quad s > 0.$$

As an application of the fractional Theta function $\theta(x, t)$, we can obtain a representation formula of the solution to (3.2) can be obtained.

Lemma 2.3. *For piecewise-continuous functions u_0 , g_0 and g_1 , the solution of*

$$\begin{cases} \sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} u - \partial_x^2 u = 0 & \text{in } (0, 1) \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } (0, 1), \\ u_x(0, \cdot) = g_0, \quad u_x(1, \cdot) = g_1 & \text{in } (0, T) \end{cases} \quad (2.4)$$

has the form

$$u = w(x, t) + v_1(x, t) + v_2(x, t), \quad (x, t) \in (0, 1) \times (0, T),$$

where

$$w(x, t) = \int_0^1 (\theta(x - \xi, t) + \theta(x + \xi, t)) u_0(\xi) d\xi,$$

$$v_1(x, t) = 2 \sum_{j=1}^{\ell} q_j \int_0^t (D_t^{1-\alpha_j} \theta)(x, t - \tau) g_0(\tau) d\tau$$

and

$$v_2(x, t) = 2 \sum_{j=1}^{\ell} q_j \int_0^t (D_t^{1-\alpha_j} \theta)(x - 1, t - \tau) g_1(\tau) d\tau.$$

The proof is extremely similar to the argument used in Rundell, Xu and Zuo [18, Lemma 3.1], so we omit the proof of this lemma.

3 Proof of the main result

3.1 Cauchy problem

Now let us turn to considering the following lateral Cauchy problem

$$\begin{cases} \sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} u = \partial_x^2 u & \text{in } (0, 1) \times (0, T), \\ u(0, \cdot) = u_x(0, \cdot) = 0 & \text{in } (0, T). \end{cases} \quad (3.1)$$

We first set $u_0(x) := u(x, 0)$ and $g(t) = u_x(1, t)$, we extend the function g to the interval $[0, +\infty)$ by letting $g \equiv 0$ outside of $(0, T)$, and by \tilde{g} we denote the extension, and by \tilde{u} we denote the solution to the following auxiliary system

$$\begin{cases} \sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} \tilde{u} - \partial_x^2 \tilde{u} = 0 & \text{in } (0, 1) \times (0, +\infty), \\ \tilde{u}(\cdot, 0) = u_0 & \text{in } (0, 1), \\ \tilde{u}_x(0, \cdot) = 0, \tilde{u}_x(1, \cdot) = \tilde{g} & \text{in } (0, +\infty). \end{cases} \quad (3.2)$$

Remark 3.1. From the estimates for the fundamental solution in Lemma 2.2, following the argument used in Li and Yamamoto [10, Lemma 2.2], one can easily show that there exist positive constants C and M such that

$$|\tilde{u}(0, t)| \leq C e^{Mt}, \quad t > 0. \quad (3.3)$$

We will prove that the boundary $\tilde{u}(0, \cdot) = 0$ in $(0, T)$ combined with (3.2) can uniquely determine the initial and boundary value, that is, $u_0 = 0$ and $\tilde{g} = 0$, which further implies the uniqueness of the lateral Cauchy problem (3.1).

Lemma 3.1. *Let $T > 0$ be a fixed constant and $u \in L^2(0, T; H^2(0, 1))$ be a solution to the lateral Cauchy problem (3.1). Then we have*

$$u(x, t) = 0, \quad (x, t) \in [0, 1] \times [0, T].$$

Proof. From the above calculations and settings, and noting lemma, we see that \tilde{u} is an extension of u which solves the Cauchy problem (3.1), that is $\tilde{u} = u$ in $[0, 1] \times [0, T]$. Using the assumption that $u(0, t) = 0$ for $t \in [0, T]$, we find

$$2 \int_0^1 \theta(\xi, t) u_0(\xi) d\xi + 2 \sum_{j=1}^{\ell} q_j \int_0^t D_t^{1-\alpha_j} \theta(1, t-\tau) \tilde{g}(\tau) d\tau = \begin{cases} 0, & t \in (0, T), \\ \tilde{u}(0, t), & t \in [T, +\infty). \end{cases}$$

Taking the Laplace transforms on both sides of the above equation, we have

$$2 \int_0^1 \mathcal{L}\{\theta(\xi, t); s\} u_0(\xi) d\xi + 2 \sum_{j=1}^{\ell} q_j \mathcal{L}\{D_t^{1-\alpha_j} \theta(1, t); s\} \mathcal{L}\{\tilde{g}(t); s\} = \int_T^{\infty} \tilde{u}(0, t) e^{-st} dt, \quad (3.4)$$

where $\mathcal{L}\{\varphi; s\}$ means the Laplace transform of a function φ .

Combining the above calculation, form (3.4), we find that

$$\begin{aligned} & \mathcal{L}\{\tilde{g}(t); s\} \\ &= \frac{1}{2} \left(\int_T^{\infty} \tilde{u}(0, t) e^{-st} dt - \int_0^1 \frac{\phi^{\frac{1}{2}}(s)}{s} \frac{e^{\xi\phi^{\frac{1}{2}}(s)} + e^{(2-\xi)\phi^{\frac{1}{2}}(s)}}{e^{2\phi^{\frac{1}{2}}(s)} - 1} u_0(\xi) d\xi \right) / \left(\sum_{j=1}^{\ell} q_j \frac{\phi^{\frac{1}{2}}(s)}{s^{\alpha_j}} \frac{e^{\phi^{\frac{1}{2}}(s)}}{e^{2\phi^{\frac{1}{2}}(s)} - 1} \right) \\ &= \frac{1}{2} \left(\sum_{j=1}^{\ell} q_j s^{-\alpha_j} \right)^{-1} \phi^{-\frac{1}{2}}(s) \left[e^{\phi^{\frac{1}{2}}(s)} - e^{-\phi^{\frac{1}{2}}(s)} \right] \int_T^{\infty} \tilde{u}(0, t) e^{-st} dt \\ & \quad - \frac{1}{2s} \left(\sum_{j=1}^{\ell} q_j s^{-\alpha_j} \right)^{-1} \int_0^1 e^{(\xi-1)\phi^{\frac{1}{2}}(s)} u_0(\xi) d\xi - \frac{1}{2s} \left(\sum_{j=1}^{\ell} q_j s^{-\alpha_j} \right)^{-1} \int_0^1 e^{(1-\xi)\phi^{\frac{1}{2}}(s)} u_0(\xi) d\xi \\ &=: I_1(s) + I_2(s) - I_3(s), \quad s > 0. \end{aligned}$$

From the choice of \tilde{g} , we see that

$$\mathcal{L}\{\tilde{g}(t); s\} = \int_0^T g(t) e^{-st} dt \leq \|g\|_{L^\infty(0, T)} \int_0^T e^{-st} dt \leq \|g\|_{L^\infty(0, T)} s^{-1}, \quad s > 0.$$

Moreover, from the estimate (3.3), it follows that

$$\int_T^{\infty} |\tilde{u}(0, t)| e^{-st} dt \leq \int_T^M C e^{(M-s)t} dt = \frac{C e^{MT}}{s-M} e^{-sT}, \quad s > 2M.$$

Then for I_1 , we have

$$\begin{aligned}
|I_1(s)| &\leq \frac{1}{2} \left(\sum_{j=1}^{\ell} q_j s^{-\alpha_j} \right)^{-1} \phi^{-\frac{1}{2}}(s) \left(e^{\phi^{\frac{1}{2}}(s)} - e^{-\phi^{\frac{1}{2}}(s)} \right) \frac{C e^{MT}}{s - M} e^{-sT} \\
&= \frac{1}{2} \frac{\left(\sum_{j=1}^{\ell} q_j s^{-\alpha_j} \right)^{-1}}{\left(\sum_{j=1}^{\ell} q_j s^{\alpha_j} \right)^{\frac{1}{2}}} \left(e^{(\sum_{j=1}^{\ell} q_j s^{\alpha_j})^{\frac{1}{2}}} - e^{-(\sum_{j=1}^{\ell} q_j s^{\alpha_j})^{\frac{1}{2}}} \right) \frac{C e^{MT}}{s - M} e^{-sT} \\
&\leq C e^{-C_1 s}, \quad s > 2M.
\end{aligned}$$

For I_2 , since $u_0 := u(\cdot, 0) \in C[0, 1]$, we have

$$\begin{aligned}
|I_2(s)| &\leq \frac{1}{2} s^{-1} \left(\sum_{j=1}^{\ell} q_j s^{-\alpha_j} \right)^{-1} \|u_0\|_{L^\infty(0,1)} \int_0^1 e^{(\xi-1)\phi^{\frac{1}{2}}(s)} d\xi \\
&\leq \frac{1}{2} \|u_0\|_{L^\infty(0,1)} \frac{1}{\phi^{\frac{1}{2}}(s)} (1 - e^{-\phi^{\frac{1}{2}}(s)}) \\
&\leq \frac{1}{2} \frac{1}{\phi^{\frac{1}{2}}(s)} \|u_0\|_{L^\infty(0,1)} \leq \frac{1}{2} \|u_0\|_{L^\infty(0,T)} s^{-\frac{\alpha_\ell}{2}}, \quad s > 0.
\end{aligned}$$

We then have

$$|I_3(s)| \leq c e^{-c_1 s} + \|g\|_{L^\infty(0,T)} s^{-1} + \frac{1}{2} \|u_0\|_{L^\infty(0,1)} s^{-\frac{\alpha_\ell}{2}}, \quad s > 2M,$$

which further implies

$$\left| \int_0^1 e^{(1-\xi)\phi^{\frac{1}{2}}(s)} u_0(\xi) d\xi \right| \leq c_2 (1 + s^{1-\frac{\alpha_\ell}{2}}), \quad s > 2M.$$

The change of variables implies

$$\left| \int_0^1 e^{\eta z} u_0(1 - \eta) d\eta \right| \leq c_2 z^\gamma, \quad \gamma \text{ large enough, } z > 2M.$$

For $0 < |z| < 2M$, we have

$$\left| \int_0^1 e^{\eta z} u_0(1 - \eta) d\eta \right| \leq \|u_0\|_{L^\infty(0,1)} e^z \leq \|u_0\|_{L^\infty(0,1)} e^{2M}.$$

Therefore

$$\left| \int_0^1 e^{\eta z} u_0(1 - \eta) d\eta \right| \leq c_3 e^{az}, \quad z > 0.$$

From Phragmén-Lindelöf principle, following the argument used in [10, Corollary 2.1], we must have

$$u_0 \equiv 0 \quad \text{in } [0, 1].$$

We next use Titchmarsh convolution theorem (see Doss [4] and Titchmarsh [21]) to derive that $\tilde{g}(t) \equiv 0$, $t > 0$, then $g \equiv 0$. From the uniqueness of the initial-boundary value problem (3.2), we see that $\tilde{u} \equiv 0$, that is $u \equiv 0$, we finish the proof of the lemma. \square

3.2 Proof of the main result

Theorem 1 directly follows from the above lemmas.

Proof of Theorem 1. Indeed, setting $I = (a, b)$ with $(a, b) \subset [0, 1]$, by $u|_{I \times (0, T)}$, we have $u(a, \cdot) = u_x(a, \cdot) = 0$ and $u(b, \cdot) = u_x(b, \cdot) = 0$ in $(0, T)$, changing independent variables $x \rightarrow a - x$ and $x \rightarrow x - b$ in the intervals $(0, a)$ and $(b, 1)$ respectively, and applying lemma 3.1, we obtain $u \equiv 0$ in $(0, a) \times (0, T)$ and $(b, 1) \times (0, T)$. \square

4 Numerical simulation

On account of the theoretical uniqueness result explained in the previous section, this section is primarily intended to develop an effective numerical method. that is, the numerical reconstruction of the solution in the domain $(0, 1) \times (0, T)$ from the addition data u in $I \times (0, T)$, where I is a subinterval of $(0, 1)$.

4.1 Iterative thresholding algorithm

We consider the initial-boundary value problem for a single-term time-fractional diffusion equation with homogeneous Dirichlet boundary condition

$$\begin{cases} \partial_t^\alpha u - \Delta u = 0, & (x, t) \in (0, 1) \times (0, T), \\ u(x, 0) = a(x), & x \in (0, 1), \\ u(0, t) = u(1, t) = 0, & t \in (0, T). \end{cases} \quad (4.1)$$

In order to emphasize that the solution of problem (4.1) depends on the unknown function a , we write it as $u(a)$.

Lemma 4.1. [5, Lemma 3.4] For $\alpha > 0$ and $g_1, g_2 \in L^2(0, T)$, there holds

$$\int_0^T (J_{0+}^\alpha g_1(t)) g_2(t) dt = \int_0^T g_1(t) J_{T-}^\alpha g_2(t) dt,$$

where $J_{T-}^\alpha g$ denotes the α -th order backward integrals of g are defined by

$$J_{T-}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (\tau - t)^{\alpha-1} g(\tau) d\tau, \quad t \in (0, T).$$

From this, we set $a_{true} \in L^2(0, 1)$ as the true solution to problem (4.1), and by using noise contaminated observation data u^δ in $I \times (0, T)$, we carry out numerical reconstruction. Where u^δ satisfies $\|u^\delta - u(a_{true})\|_{L^2(I \times (0, T))} \leq \delta$ and δ stands for the noise level. For avoiding ambiguity, we specify $u^\delta = 0$ out of $I \times (0, T)$ so that it is well-defined in $(0, 1) \times (0, T)$.

By using the Tikhonov regularization technique, we can transform the reconstruction of the initial value into the minimization of the following output least squares functional

$$\min_{a \in L^2(0,1)} \Phi(a), \quad \Phi(a) := \|u(a) - u^\delta\|_{L^2(I \times (0,T))}^2 + \rho \|a\|_{L^2(0,1)}^2, \quad (4.2)$$

where $\rho > 0$ is the regularization parameter.

Now we need the information about the Fréchet derivative $\Phi'(a)$ of the objective functional $\Phi(a)$. For an arbitrarily fixed direction $g \in L^2(0,1)$, it follows from direct calculations that

$$\begin{aligned} \Phi'(a)g &= 2 \int_0^T \int_I (u(a) - u^\delta)(u'(a)g) dxdt + 2\rho \int_0^1 agdx \\ &= 2 \int_0^T \int_I (u(a) - u^\delta)u(g) dxdt + 2\rho \int_0^1 agdx. \end{aligned} \quad (4.3)$$

Here $u'(a)g$ denotes the Fréchet derivative of $u(a)$ in the direction g , and the linearity of (4.1) immediately yields

$$u'(a)g = \lim_{\epsilon \rightarrow 0} \frac{u(a + \epsilon g) - u(a)}{\epsilon} = u(g).$$

It is clear that using (4.3) to evaluate $\Phi'(a)g$ for all $g \in L^2(0,1)$ is extremely expensive, since one should solve system (4.1) for $u(g)$ with g varying in $L^2(0,1)$ in the computation for a fixed a . We introduce the adjoint system of (4.1) to reduce the computational costs for the Fréchet derivatives, that is, the following system for a backward time-fractional diffusion equation

$$\begin{cases} D_T^\alpha v + \Delta v = \chi_I(u(a) - u^\delta), & (x, t) \in (0, 1) \times (0, T), \\ J_T^{1-\alpha} v(x, T) = 0, & x \in (0, 1), \\ v(0, t) = 0, v(1, t) = 0, & t \in (0, T), \end{cases} \quad (4.4)$$

where $D_T^\alpha u$ stands for the backward Riemann-Liouville derivative of u which is defined by $D_T^\alpha u := \partial_t J_T^{1-\alpha} u$, and χ_I denotes the characterization function of I , and we write the solution of (4.4) as $v(a)$.

We can further treat the first term in (4.3) as

$$\begin{aligned} \int_0^T \int_I (u(a) - u^\delta)u(g) dxdt &= \int_0^T \int_0^1 \chi_I(u(a) - u^\delta)u(g) dxdt \\ &= \int_0^T \int_0^1 (D_T^\alpha v + \Delta v)u(g) dxdt \\ &= - \int_0^1 J_T^{1-\alpha} v_a(0)g dx, \end{aligned}$$

implying

$$\Phi'(a)g = 2 \int_0^1 (\rho ag - J_T^{1-\alpha} v_a(0)g) dx, \quad \forall g \in L^2(0,1).$$

This suggests a characterization of the solution to the minimization problem (4.2).

Lemma 4.2. *The function $a^* \in L^2(0,1)$ is a minimizer of the functional $\Phi(a)$ in (4.2) only if it satisfies the variational equation*

$$\rho a^* g - J_T^{1-\alpha} v_{a^*}(0) g = 0, \quad (4.5)$$

where $v_{a^*}(0)$ solves the backward problem (4.4) with the coefficient a^* .

We can obtain the iterative thresholding algorithm by adding $Ma^*(M > 0)$ to both sides of (4.5) and rearranging the equation,

$$\begin{aligned} a_{k+1} &= \frac{M}{M+\rho} a_k + \frac{1}{M+\rho} J_T^{1-\alpha} v_{a_k}(0) \\ &= \frac{M}{M+\rho} a_k + \frac{1}{M+\rho} \frac{1}{\Gamma(1-\alpha)} \int_t^T (\tau-t)^{-\alpha} v_{a_k}(x, \tau) d\tau|_{t=0} \\ &= \frac{M}{M+\rho} a_k + \frac{1}{M+\rho} \frac{1}{\Gamma(1-\alpha)} \int_0^T \tau^{-\alpha} v_{a_k}(x, \tau) d\tau, \end{aligned} \quad (4.6)$$

Where $M > 0$ is a tuning parameter for the convergence, it suffices to choose

$$M \geq \|A\|_{op}^2, \quad \text{where}$$

$$\begin{aligned} A : L^2(0,1) &\rightarrow L^2(I \times (0,T)), \\ a &\mapsto u(a)|_{I \times (0,T)}. \end{aligned} \quad (4.7)$$

Here $\|\cdot\|_{op}$ denotes the operator norm of an operator under consideration. There is the iterative thresholding algorithm for the reconstruction of the initial value.

Algorithm 4.1. *Choose a tolerance $\varepsilon > 0$, a regularization parameter $\rho > 0$ and a tuning constant $M > 0$ according to (4.7). Give an initial guess $a_0 \in L^2(0,1)$, and set $k = 0$.*

1. Compute a_{k+1} by the iterative update (4.6).

2. If $\|a_{k+1} - a_k\|_{L^2(0,1)} / \|a_k\|_{L^2(0,1)} < \varepsilon$, stop the iteration. Otherwise, update $k \leftarrow k + 1$ and return to Step 1.

As can be seen from (4.6), we only need to solve the forward problem (4.1) once for $u(a_k)$ and the backward problem (4.4) once for $v(a_k)$ subsequently at each iteration step. Therefore, the numerical implementation of Algorithm 1 is easy and computationally cheap.

As a result, for the adjoint system of (4.4), we know that $v(a)$ is the solution to the following problem with the Caputo derivative

$$\begin{cases} -\partial_t^\alpha(v) + \Delta v = \chi_I(u(a) - u^\delta) & \text{in } (0,1) \times (0,T), \\ v = 0 & \text{in } (0,1) \times \{0\}, \\ v(0,t) = v(1,t) = 0 & \text{in } (0,T), \end{cases} \quad (4.8)$$

because of the homogeneous terminal value $J_T^{1-\alpha} v(\cdot, T) = 0$, it suffices to deal with (4.8) instead of (4.4) by the same forward solver for (4.1) in the numerical simulation.

4.2 Numerical experiments

In this section, we will apply the iterative thresholding algorithm established in the previous subsection to the numerical treatment of unique continuation of the solution, that is, the identification of the value of the solution in $(0, 1) \times (0, T)$ from the addition data $u|_{I \times (0, T)}$. We will illustrate the effectiveness of the reconstruction method through a large number of test examples.

With the true solution $a_{true} \in L^2(0, 1)$, we produce the noisy observation data u^δ by adding uniform random noises to the true data, i.e

$$u^\delta(x, t) = (1 + \delta \text{rand}(-1, 1))u(a_{true})(x, t), \quad (x, t) \in I \times (0, T).$$

Here $\text{rand}(-1, 1)$ denotes the uniformly distributed random number in $[-1, 1]$ and $\delta \geq 0$ is the noise level.

In addition to illustrative graphs, we mainly use the relative L^2 -norm error to evaluate numerical performance

$$err_a := \frac{\|a_r - a_{true}\|_{L^2(0,1)}}{\|a_{true}\|_{L^2(0,1)}}$$

where a_r is regarded as the reconstructed solution produced by Algorithm 4.1.

Then, we study the solution of the direct problem, the numerical solution obtained by difference scheme is compared with the reconstructed solution obtained by inversion algorithm, we evaluate the numerical performance by the relative L^2 -norm error

$$err_u := \frac{\|u_r - u_n\|_{L^2(0,1)}}{\|u_n\|_{L^2(0,1)}}$$

Where u_r stands for the reconstructed solution and u_n denotes the numerical solution.

We divide the space-time region $[0, 1] \times [0, 1]$ into 40×40 equidistant meshes. First we fix the noise level $\delta = 0.5\%$ and the observation subdomain $I = (0, 0.05) \cup (0.95, 1)$ and test the algorithm with the following settings:

(a) $\alpha = 0.2$, $M = 8 \times 10^{-4}$, $\rho = 10^{-7}$, $a_{true}(x) = \frac{\sin(\pi x) - x(x-1)}{\Gamma(2-\alpha)}$,

initial guess $a_0(x) = 13$, set the tolerance parameter $\varepsilon = 10^{-4}$.

(b) $\alpha = 0.4$, $M = 9 \times 10^{-4}$, $\rho = 9 \times 10^{-9}$, $a_{true}(x) = \frac{x-x^2}{\Gamma(2-\alpha)}$,

initial guess $a_0(x) = 3$, set the tolerance parameter $\varepsilon = 10^{-4}$.

(c) $\alpha = 0.3$, $M = 10^{-4}$, $\rho = 10^{-7}$, $a_{true}(x) = (x-1)(e^x - 1)$,

initial guess $a_0(x) = -8$, set the tolerance parameter $\varepsilon = 4 \times 10^{-4}$.

(d) $\alpha = 0.3$, $M = 10^{-4}$, $\rho = 10^{-7}$, $a_{true}(x) = 1 - |2x - 1|$,

initial guess $a_0(x) = -8$, set the tolerance parameter $\varepsilon = 4 \times 10^{-4}$.

In figure 1-2, for case(a), the iteration steps $k = 786$ is obtained using the Algorithm 4.1 to obtain the reconstructed solution of the initial value, and the relative error of the reconstructed solution and the true solution of the initial value $err_a = 4.71\%$, the relative error of the reconstructed solution of the direct problem obtained by inversion algorithm

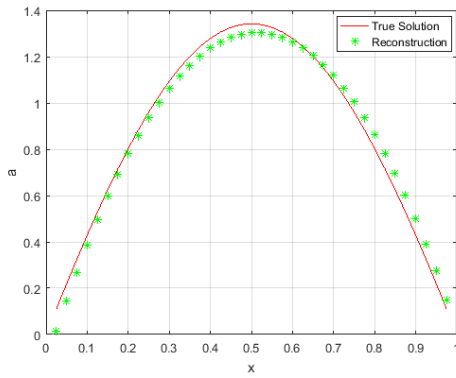


Figure 1: Case(a): True solutions a_{true}

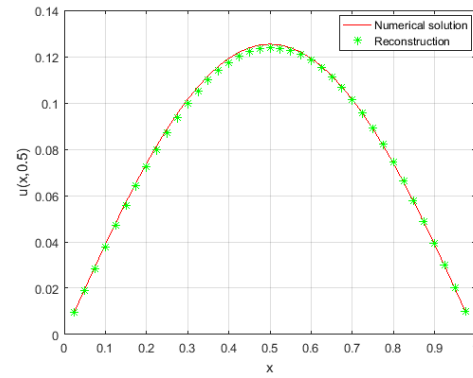


Figure 2: Case(a): Numerical solutions u_n and reconstruction u_r

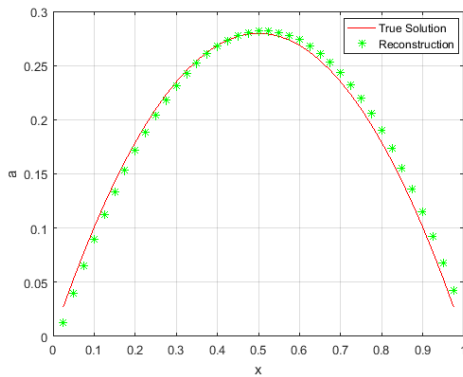


Figure 3: Case(b): True solutions a_{true}

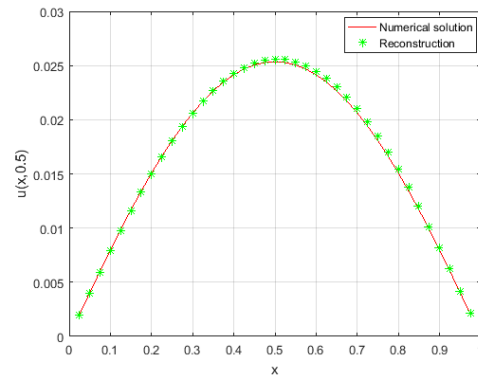


Figure 4: Case(b): Numerical solutions u_n and reconstruction u_r

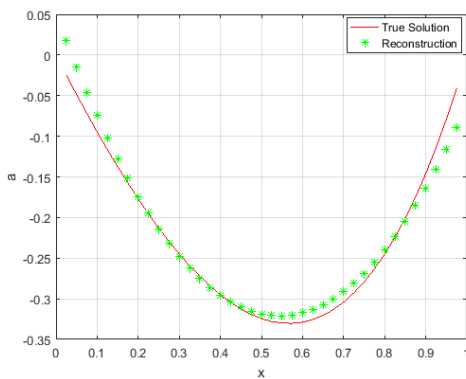


Figure 5: Case(c): True solutions a_{true}

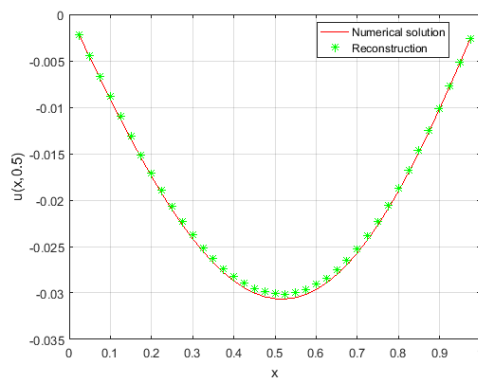


Figure 6: Case(c): Numerical solutions u_n and reconstruction u_r

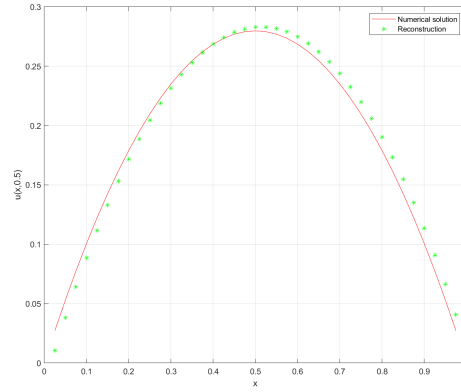
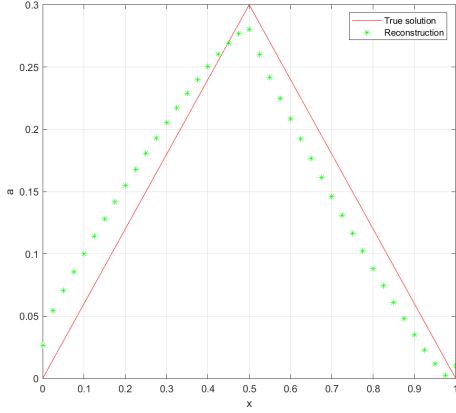


Figure 7: Case(d): True solutions a_{true} and Figure 8: Case(d): Numerical solutions u_n and reconstruction u_r

and the numerical solution obtained by difference scheme $err_u = 1.36\%$. In figure 3-4, for case(b), the iteration steps $k = 863$ and the relative error between the reconstructed solution and the true solution of the initial value $err_a = 4.36\%$, the relative error of the numerical solution and the reconstructed solution of the direct problem $err_u = 1.29\%$. In figure 5-6, for case(c), the iteration steps $k = 1016$ and the relative error between the reconstructed solution and the true solution of the initial value $err_a = 6.80\%$, the relative error of the numerical solution and the reconstructed solution of the direct problem $err_u = 1.77\%$. In figure 7-8, for case(d), owing to the poor regularity of the target function, the reconstruction results are not as well as those of the smooth case in the previous examples. The iteration steps $k = 2217$ and the relative error between the reconstructed solution and the true solution of the initial value $err_a = 12.67\%$, the relative error of the numerical solution and the reconstructed solution of the direct problem $err_u = 3.56\%$. Figure 1-6 and the relative error err_a, err_u indicate the efficiency and accuracy of the proposed Algorithm 4.1 for simulating the unique continuation.

5 Concluding remarks

In this paper, on the basis of the Theta function method, we first gave a representation formula of the solution and showed the uniqueness of the solution to the Cauchy problem by the use of the Laplace transform argument, from which we further verified the unique continuation property of the solution to (1.1). Specifically, we proved that a solution to a fractional diffusion equation can be uniquely determined from an interior subdomain. The method used in the proof is new method that can effectively overcome the difficulties caused by the non-locality of fractional derivatives. This result and the used methods not only provide theoretical foundation for the fractional diffusion equations but also open up new possibilities for studying related inverse problems in areas such as inverse source problems and approximate control problem in practical applications.

Let us mention that the proof of the unique continuation principle heavily relies on the

Theta function method which enables one to derive an explicit representation formula of the solution. It would be interesting to investigate what happens about the unique continuation property of the solution in the general dimensional case. Moreover, in Theorem 1, we only proved the uniqueness result for determining the solution from the information of the solution in interior domain. In comparison, it is known that conditional stability results hold for the same problems for elliptic or parabolic equations based on Carleman estimates. Unfortunately, such techniques do not work in the case of fractional diffusion equations due to the absence of the fundamental integration by parts for the fractional derivatives.

In the numerical aspect, we reformulate the unique continuation principle as an optimization problem with Tikhonov regularization. After the derivation of the corresponding variational equation, we can characterize the minimizer by employing the associated backward fractional diffusion equation, which results in our iterative method. Then several numerical experiments for the reconstructions are implemented to show the efficiency and accuracy of the proposed Algorithm 4.1 for simulating the unique continuation. Here we point out that the homogeneous boundary condition was assumed in deriving Algorithm 4.1. It will be intriguing to derive Algorithm 4.1 without assuming the homogeneous boundary condition, as the algorithm for the general case still remains a challenge to be solved.

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