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Contributions to Pure and Applied Mathematics

On some extremes of mixing for Copula-based Markov chains

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Abstract

This paper presents a new general construction of copula that includes some known families such as the Farlie-Gumbel-Morgenstern copula family. This general form of copula helps address extreme cases of mixing and justifies optimality of the results of Longla [1] and Longla [2] on mixing for copula-based Markov chains. Some examples are presented to show that the results can not be extended by weakening the assumptions.
keywords Copula-based Markov chains, Mixing for Markov chains, ergodicity, Markov chain central limit theorem

1 Introduction

In this paper, a copula is a bivariate function $C(u,v)$ defined on $[0,1]^2$, such that $C(0,u) = C(u,0) = 0$ and $C(u,1) = C(1,u) = u$ for all $u \in [0,1]$ and

$$C(u_1,v_1) + C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) \geq 0, \quad \text{for all} \quad (u,v) \in [0,1]^2.$$

This definition is equivalent in probability theory to the the requirement that $C(u,v)$ is the joint cumulative distribution of two random variables, each of which has a uniform distribution on $[0,1]$ [1]. Darsow et al [2] present properties of such functions and their relationship to Markov chains. Namely, a one step discrete time stationary Markov chain, can be represented by its copula and stationary distribution. To obtain a non-stationary Markov chain, one can modify the copulas of some consecutive variables of the chain or some marginal distributions. Moreover, if the chain is stationary with uniform marginal distribution, then the transition probabilities are fully defined by $C_{,1}(u,v)$, which is the derivative of $C(u,v)$ with respect to its first parameter. Among important properties of a copula, we have absolute continuity, which allows to find probabilities using integrals of the density function. We say that a copula is absolutely continuous if there exists a function $c(u,v)$ defined on $[0,1]^2$ such that for all $(u,v) \in (0,1)^2$,

$$C(u,v) = \int_0^u \int_0^v c(s,t)dtds = AC(u,v), \quad c(u,v) = \frac{\partial^2}{\partial u \partial v} C(u,v).$$

The function $c(u,v)$ is called density of the copula $C(u,v)$. In general, a copula has an absolutely continuous part denoted $AC(u,v)$. And $C(u,v) = AC(u,v)$ if and only if the copula is absolutely continuous. This class of copulas has been widely studied for mixing properties in several works such as [3-5] and the references therein. The singular part of the copula is defined as $SC(u,v) = C(u,v) - AC(u,v)$. Some copulas, such as the Hoeffding lower and upper bounds are singular, which means that their absolutely continuous part is equal to 0. Copulas such as Clayton, Farlie-Gumbel-Morgenstern (FGM), Gaussian, trigonometric copulas of Chesneau [6] are absolutely continuous. Copulas of the Mardia or Frechet families have non-zero absolutely continuous and singular parts. The copulas that we consider in this work are absolutely continuous. The fold product of copulas that is used to find the joint distribution of (U_0, U_n) is given by $C^1(u,v) = C(u,v)$ and $C^{n+1}(u,v) = C^n * C(u,v)$ for $n > 1$; where $A * B(u,v) = \int_0^1 A_{,2}(u,t)B_{,1}(t,v)dt$ and $f_{,i}$ is the derivative with respect to the i-th variable of f.

2 Copula models and mixing

Here we introduce the mixing property that is used to establish the central limit theorem for parameter estimators of copula models based on our model. We say that the Markov chain generated by a copula $C(u,v)$ is ψ-mixing, if $\psi_n(C) \to 0$ as $n \to \infty$, where $\psi_n(C)$ is defined as follows. Let $A, B \subset \mathbb{I} = [0,1]$. Assume that $\mu^n(A \times B)$ is the measure of the set $A \times B$ induced by the joint distribution of any of its consecutive states. Denote $\mu(A)$ the Lebesgue measure of A. Assume that the Marginal distribution of the Markov chain is uniform on $[0,1]$. By Longla [5], $\mu^n(A \times B) = P(X_0 \in A, X_n \in B)$, and

$$\psi_n(C) = \sup_{A,B} \left| \frac{\mu^n(A \times B)}{\mu(A)\mu(B)} - 1 \right|.$$

This coefficient allows to control the variance of partial sums of functions of the Markov chains when establishing the central limit theorem. Longla [4] provided a result for convex combinations of stationary Markov chains, wile Longla et al. [5] provided a result for Markov chains generated by absolutely a continuous copula of bounded density. Bradley [7] presented a survey on the topic of strong mixing conditions, including ψ-mixing. Several authors have worked on mixing coefficients and their implications for the central limit theorem. Ibragimov [8], proposed a central limit theorem implied by some rates of mixing. This last result is applicable to the Markov chains that we study in this paper, because ψ-mixing implies ϕ-mixing; and the properties of our copulas helps compute in closed form the variance of partial sums.

2.1 Our copula examples

Consider a bounded function $\varphi(x)$: $\int_0^1 \varphi(x)dx = \int_0^1 \varphi^2(x)dx - 1 = 0$. Define

$$C_\lambda(u,v) = uv + \lambda\Phi(u)\Phi(v), \quad \Phi(u) = \int_0^u \varphi(s)ds. \tag{2.1}$$

Theorem 2.1. *The Function $C_\lambda(u,v)$ is a copula if and only if $-\frac{1}{\max\varphi^2(x)} \le \lambda \le -\frac{1}{\min\varphi(x)\max\varphi(x)}$. Moreover, a Markov chain $(U_0,\cdots,U_n)$ generated by (2.1) with $|\lambda|< 1$, is ψ-mixing and the joint distribution of (U_0,U_n) is $C^n_\lambda(u,v) = C_{\lambda^n}(u,v)$.*

Theorem 2.1 shows that along the Markov chain, the joint distribution of any two variables remains in the copula family. This fact eases the study of mixing properties and large sample theory for functions of the Markov chain.

Longla et al [5] showed that this mixing coefficient converges to 0 if the copula is absolutely continuous, the marginal distribution is continuous and for some integer m, the density of the copula $C^m(u,v)$, is strictly bounded by 2 ($c^m(u,v) < 2$ for all $u,v \in [0,1]$). Using the distribution of (U_0,U_m), for any integer m, the density of $C^m_\lambda(u,v)$ satisfies $|c^m_\lambda(u,v)-1|< k|\lambda|^m$, where k is the smallest number larger than $\varphi^2(x)$ for all $(u,v)\in[0,1]^2$. Given that $|\lambda|< 1$, it follows that $c^m(u,v) < 2$ for large enough values of m. Therefore, by Longla et al [5], Markov chains generated by $C_\lambda(u,v)$ and any continuous distribution is ψ-mixing.

2.2 Extremes of the copula family

Consider $\varphi(x) = \sqrt{\alpha}\mathbb{I}(x < \frac{1}{1+\alpha}) - \frac{1}{\sqrt{\alpha}}\mathbb{I}(x \ge \frac{1}{1+\alpha})$, $0 < \alpha \le \infty$ and $\min(\alpha, 1/\alpha) \le \lambda < 1$. Let $I = [0, \frac{1}{1+\alpha}]^2$, $II = [0, \frac{1}{1+\alpha}] \times (\frac{1}{1+\alpha}, 1]$, $III = (\frac{1}{1+\alpha}, 1] \times [0, \frac{1}{1+\alpha}]$ and $IV = (\frac{1}{1+\alpha}, 1]^2$.

1. If $\lambda = -1/\alpha$, $1 < \alpha < \infty$, then the density of the copula is $c_\lambda(u,v) = \frac{(1+\alpha)}{\alpha}\mathbb{I}((u,v) \in II \cup III) + \frac{(\alpha^2-1)}{\alpha^2}\mathbb{I}((u,v) \in IV)$. This copula density is equal to zero on the set of non-zero mesure Lebesgue measure I, but generates ψ-mixing Markov chains.
2. If $\lambda = -\alpha$, $0 < \alpha < 1$, then the density of the copula is $c_\lambda(u,v) = (1+\alpha)\mathbb{I}((u,v) \in II \cup III) + (1-\alpha^2)\mathbb{I}((u,v) \in I)$. This copula density is equal to zero on the set of non-zero Lebesgue measure IV, but generates ψ-mixing Markov chains.
3. If $\lambda = 1$, $0 < \alpha < \infty$, then the density of the copula is $c_\lambda(u,v) = (1+\alpha)\mathbb{I}((u,v) \in I) + (\alpha^{-1}+1)\mathbb{I}((u,v) \in IV)$. This copula density is equal to zero on the set of non-zero Lebesgue measure $II \cup III$, and does not generate ergodic Markov chains. This density is not strictly less than 2.
4. For $\alpha = 1$ and $\lambda = 1$ (or $\lambda = -1$), we have a density that is constant and equal to 2 on its support. This example shows that the result of Longla et al. [5] can not be extended by relaxing the condition $c^m(u,v) < 2$ for some integer m.

3 conclusion

This example shows that the results of Longla [5] cannot be extended in general. There exist ψ-mixing Markov chains based on copulas with density equal to 0 on non-zero measure sets.

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