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Singular Integral Equations on a Set of Distributions

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Abstract

Let D be a connected bounded domain in $\mathbb{R}^2$, S be its boundary which is closed, connected and smooth, or $S=(-\infty,\infty)$. Let $f=B\phi=\frac{1}{\pi i}\int_S\frac{\phi(s)ds}{s-t}ds$. Here $\phi,f\in X$, where X is a set of distributions, that is, ϕ,f are bounded linear functionals on the Banach space H^μ of Hölder-continuous functions on S. A definition of the operator B on X is given. The equation $B\phi=f$ is studied. It is proved that this equation has a solution in X, and this solution is unique in X.

1 Introduction

Let D be a connected bounded domain on $\mathbb{R}^2$, S be its boundary, which is closed and $C^{1,a}$-smooth, $0<a\le 1$ or $S=(-\infty,\infty)$. Consider the singular integral equation (see [1]):

$$f=B\phi=\frac{1}{i\pi}\int_S\frac{\phi(s)}{s-t}ds. \tag{1.1}$$

We assume that $f\in X$. This is *the basic new assumption*. Our goal is to give a *new definition* of the operator B. This definition makes the proof of the existence of the unique solution to the equation $B\phi=f$ in X very simple.

MSC: 45E05
Key words: singular integral equations; new definition of singular integral operator

Let $H^\mu := H^\mu(S)$ be the space of Hölder-continuous functions, $0 < \mu < 1$, with the usual norm, and X be the Banach space of the bounded linear functionals h on H^μ with the standard norm:

$$\|h\|_X = \sup_{\|\psi\|_{H^\mu}=1} |h(\psi)|, \quad \psi \in H^\mu. \tag{1.2}$$

One has

$$|h(\psi)| \leq \|h\|_X \|\psi\|_{H^\mu}. \tag{1.3}$$

It is known, see [1], that on H^μ the operator B is bounded, has a bounded inverse, the range $R(B)$ of B is equal to H^μ, $R(B) = H^\mu$, and $B^2 = I$, where I is the identity operator. Therefore, the equation

$$B\phi = f \tag{1.4}$$

is solvable in H^μ for any $f \in H^\mu$ and its solution $\phi = Bf$ is unique. One has $\|B\psi\|_{H^\mu} \leq c_\mu \|\psi\|_{H^\mu}$, where c_μ is a constant.

The aim of this paper is to generalize this result to the space X which contains distributions. By $D(B)$ and $R(B)$ the domain of definition of B and, respectively, the range of B are denoted. Our result is the following Theorem.

Theorem 1. *The $D(B) = X$, $R(B) = X$, the operators B and B^{-1} are bounded.*

Theorem 1 is proved in Section 2.

2 Proofs

Let us define B on X:

Definition 1.

$$(B\phi, \psi) = -(\phi, B\psi) \quad \forall \psi \in H^\mu(S),\ 0 < \mu < 1. \tag{2.1}$$

Here (h, ψ) is the value of the functional $h \in X$ on the element $\psi \in H^\mu$. This definition is similar to the one used in [3]. We prove that B is defined on all of X and is bounded on X.

Lemma 1. *The operator $B : X \to X$ is bounded, $D(B) = X$.*

Proof. Let $\phi \in X$. Then, using Definition 1, one gets:

$$|(B\phi, \psi)| = |(\phi, B\psi)| \leq \|\phi\|_X \|B\psi\| \leq c_\mu \|\phi\|_X \|\psi\|_{H^\mu}, \quad \forall \psi \in H^\mu(S),\ 0 < \mu < 1. \tag{2.2}$$

Thus, $D(B) = X$ and $||B||_X \leq c_\mu$.

Lemma 1 is proved. □

Lemma 2. *Relation $B^2 = I$ holds in X.*

Proof. By definition (2.1) one gets:

$$(B^2\phi, \psi) = -(B\phi, B\psi) = (\phi, B^2\psi) = (\phi, \psi) \quad \forall \psi \in H^\mu, \tag{2.3}$$

where the last equation holds because $B^2 = I$ on H^μ.

Therefore, Lemma 2 is proved. □

From equation (1.4) and from Lemma 2 one gets:

$$\phi = Bf. \tag{2.4}$$

Lemma 3. *Equation* (1.4) *has a solution in X and this solution is unique.*

Proof. First, let us prove the uniqueness. If $B\phi_1 = f = B\phi_2$, then $B\phi = 0$, where $\phi := \phi_1 - \phi_2$. By (2.1), one has $(\phi, B\psi) = 0, \forall\psi \in H^\mu$. Since the set $B\psi = H^\mu$ when ψ runs through all of H^μ, it follows that $\phi = 0$. The uniqueness of the solution in X is proved.

Let us prove the existence of the solution. By Lemma 2 and equation (2.4), one obtains:

$$B^2\phi = \phi = Bf. \tag{2.5}$$

Therefore, Lemma 3 is proved. □

The conclusions of Theorem 1 follows from Lemmas 1–3. □

Remark 1. *It does not follow that B maps $L^1(S)$ into itself.*

Example 1. Let us show that there is an $f \in L^1(S)$ such that $Bf \notin L^1(S)$. Let $S = (-\infty, \infty)$, $\mathcal{F}(f) := \tilde{f}$, $\tilde{f} := \int_S e^{i\xi s} f(s)ds$, $\mathcal{F}(Bf) = \mathcal{F}(f)\mathcal{F}(s^{-1}) = \tilde{f} i\pi sgn(\xi)$. We have used the known formula, see [2]: $\mathcal{F}(s^{-1}) = i\pi sgn(\xi)$, where $sgn(\xi) = 1$ if $\xi > 0$, $sgn(\xi) = -1$ if $\xi < 0$. The Fourier transform of $f \in L^1(S)$ is a continuous uniformly bounded function. Therefore, $\tilde{f} sgn(\xi)$ is not, in general, a continuous function at $\xi = 0$. Thus, if $\tilde{f}|_{\xi=0} \neq 0$, then the function $Af \notin L^1(S)$.

3 Conclusion

A new definition of the singular integral operator in a special space X of distributions is given.

4 Conflict of interest

There is no conflict of interest.

References

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