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On Complete Positivity of Sparse Symmetric Toeplitz Matrices

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Abstract. Completely positive matrices play a central role in optimization and mathematical modeling, particularly in reformulating NP-hard problems as linear programs over convex cones. This paper investigates the complete positivity of symmetric Toeplitz matrices with two nonzero coefficients, a sparse and structured class of matrices. By exploring their connection to Bernoulli-compatible matrices, we derive novel sufficient conditions for their complete positivity and extend these results to specific probabilistic and optimization frameworks. The findings deepen our understanding of structured matrices and enhance their applicability in stochastic modeling, control theory, and signal processing. Additionally, we provide illustrative examples and practical implications for these conditions, contributing to both theoretical advancements and computational feasibility in matrix analysis.

1. Introduction

Completely Positive Matrices (CPMs) hold a central position in mathematical research, connecting various fields ranging from operator theory and harmonic analysis to their practical applications in computer vision and signal processing. These matrices not only deepen our comprehension of positive semidefinite matrices but also provide valuable insights into numerous areas of mathematics and engineering. Notably, in the realm of optimization, a substantial number of NP-hard optimization problems defined over convex cones can be reformulated as linear optimization problems over the cone of completely positive matrices, see [2], [5], [8], [11], and [15].

A CPM is a real $d \times d$ square matrix $\mathbf{A}$ that admits a factorization

$$\mathbf{A} = \mathbf{B}\mathbf{B}^{\mathsf{T}},$$

where $\mathbf{B}$ is an $d \times m$ matrix with nonnegative elements. This factorization highlights the intrinsic positive semidefinite and symmetry aspects of such matrices. We denote the set of all $d \times d$ completely positive matrices by $\mathcal{C}_d$. Although determining complete positivity is NP-hard (see [9]), we identify some subsets of $\mathcal{C}_d$ that can be easily determined.

This paper focuses on symmetric Toeplitz matrices, a special class of matrices where each descending diagonal from left to right is constant, as described by:

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{d-2} & \alpha_{d-1} \\ \alpha_1 & \alpha_0 & \alpha_1 & \ddots & & \alpha_{d-2} \\ \alpha_2 & \alpha_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \alpha_1 & \alpha_2 \\ \alpha_{d-2} & & \ddots & \alpha_1 & \alpha_0 & \alpha_1 \\ \alpha_{d-1} & \alpha_{d-2} & \cdots & \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix}, \tag{1.1}$$

where $\alpha_i \geq 0$, for $i = 0, \ldots, d-1$. Complete positivity is vital in the realm of Toeplitz matrices, enhancing our theoretical understanding of these matrices and bolstering their practical utility. This property ensures positive semidefiniteness, making Toeplitz matrices, especially the positive semidefinite ones, essential in engineering fields such as stochastic filtering ([16]), digital signal processing ([7] and [19]), and control theory ([18]). For an in-depth examination of the subject, please refer to the comprehensive review provided in [1].

Our motivation stems from the need to understand the structure and properties of these matrices, particularly their role in extreme value theory, where Bernoulli compatible matrices are a significant concept. We delve into the question of when a symmetric Toeplitz matrix can be a Bernoulli compatible matrix and extend this inquiry to the realm of complete positivity.

A similar direction but different focuses is the work in [10] which is centered on investigating the complete positivity of matrices characterized by particular sparsity patterns, including acyclic or circular matrices. It demonstrates that in such instances, the verification of complete positivity and the determination of factorizations can be achieved in linear time. For further related studies, readers are directed to the references contained therein.

The paper is organized as follows: Section 2 revisits the concept of Bernoulli compatible matrices, pivotal in extreme value theory. In Section 3, we present a sufficient condition for symmetric Toeplitz matrices with two non-zero coefficients to qualify as Bernoulli compatible matrices and explore their extension to complete positivity.

Throughout, we adhere to the convention of representing vectors and matrices in boldface, with a_{ij}, denoting the element in the i-th row and j-th column of a matrix $\mathbf{A}$. The vectors and matrices with all elements equal to 0 and 1 are denoted by $\mathbf{0}$ and $\mathbf{J}$, respectively. Our standard probability space is denoted by $([0,1], \mathcal{L}, \lambda)$, where $\mathcal{L}$ consists of the Lebesgue measurable subsets within the interval $[0,1]$, and λ is the Lebesgue measure defined on $[0,1]$. The ceiling function is denoted by $\lceil \cdot \rceil$.

2. An overview of Bernoulli compatible matrices

In this section, we explore the intricacies of Bernoulli compatible matrices (BCMs). These matrices serve as a crucial bridge between discrete random processes and matrix theory, thereby playing an indispensable role in a wide range of mathematical and statistical applications.

A matrix $\mathbf{B}$ of size $d \times d$ is classified as a Bernoulli compatible matrix if it can be represented as $\mathbb{E}(\mathbf{X}\mathbf{X}^\top)$, where $\mathbf{X}$ is a $d \times 1$ random vector taking values in $\{0,1\}^d$. The set of all such matrices is denoted by $\mathcal{B}_d$. Notably, the matrix $\mathbb{E}(\mathbf{X}\mathbf{X}^\top)$ forms a convex combination of points in the set:

$$\left\{\mathbf{x}\mathbf{x}^\top : \mathbf{x} \in \{0,1\}^d\right\}, \tag{2.1}$$

implying that $\mathcal{B}_d$ is the convex hull of these points, a closed convex set. In contrast, $\mathcal{C}_d$ represents the convex cone with extreme directions formed by $\left\{\mathbf{x}\mathbf{x}^\top : \mathbf{x} \in [0,1]^d\right\}$ (see Section 2.2 of [3]). It is important to note that $\mathcal{B}_d \subseteq \mathcal{C}_d$, and this assertion is substantiated by Corollary 2.3 in [13], which states:

Proposition 2.1. *Any Bernoulli compatible matrix is completely positive.*

Consider a matrix $\mathbf{B} \in \mathcal{B}_d$, associated with a random vector $\mathbf{X} = (X_1, \ldots, X_d)^\top$ taking values in $\{0,1\}^d$ such that $\mathbf{B} = \mathbb{E}(\mathbf{X}\mathbf{X}^\top)$. Defining events $A_i := \{X_i = 1\}$, for $i = 1, \ldots, d$, the matrix $\mathbf{B}$ can be expressed as:

$$
\mathbf{B} = \begin{bmatrix} \Pr(A_1) & \Pr(A_1 \cap A_2) & \cdots & \Pr(A_1 \cap A_d) \\ \Pr(A_1 \cap A_2) & \Pr(A_2) & \cdots & \Pr(A_2 \cap A_d) \\ \vdots & \vdots & \ddots & \vdots \\ \Pr(A_1 \cap A_d) & \Pr(A_2 \cap A_d) & \cdots & \Pr(A_d) \end{bmatrix}. \tag{2.2}
$$

This leads us to inquire whether a matrix's membership in $\mathcal{B}_d$ can be determined by the existence of events A_i in a probability space, such that their probabilities and those of their binary intersections correspond to the matrix's entries. This query is affirmatively addressed in Corollary 3 of [17]:

Proposition 2.2. *A $d \times d$ matrix $\mathbf{B}$ is a BCM if and only if there exist events A_i, $i = 1, \dots, d$, on a probability space such that the matrix $\mathbf{B}$, as defined in (2.2), is equal to $\mathbf{B}$.*

This proposition forms a cornerstone of our analysis in the subsequent section, where we explore the conditions under which a given matrix can be classified as a BCM.

3. Symmetric Toeplitz matrices with two non-zero α_is

In this section, we explore special cases of symmetric Toeplitz matrices. Building upon the work of [17], which provides sufficient and necessary conditions for 1- and 2-dependent matrices (where only α_1 is positive and only both α_1 and α_2 are positive, respectively), we extend these results to consider the cases where any two α_is are positive. This extension enables a deeper understanding of the structural nuances of symmetric Toeplitz matrices in specific sparse configurations.

We recall that $\mathcal{C}_d$ represents a convex cone, a fact that leads to an immediate and important implication for completely positive matrices, as stated in the following proposition:

Proposition 3.1. *Let $\mathbf{B}$ be any completely positive matrix. Then $k\mathbf{B}$ is completely positive for any $k \geq 0$.*

Interestingly, this property allows us to focus on symmetric Toeplitz matrices with diagonal elements normalized to $\alpha_0 := 1/d$. This representation aligns with the probability space framework and sets the stage for the ensuing analysis.

We now consider symmetric Toeplitz matrices where two of the α_i values, denoted as $\alpha_m = \alpha$ and $\alpha_{m+n} = \beta$, for $m = 1, \dots, d-2$, and $n = 1, \dots, d-m-1$, are positive, while all other α_i values are zero. In this case, the matrix takes the form:

$$
\begin{array}{c} \\ \\ 1 \\ \vdots \\ \\ m \\ \\ \vdots \\ \\ m+n \\ \\ \vdots \\ d-1 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} & 1 & \dots & & m & & \dots & & m+n & & \cdots & d-1 \end{array} \\
\left[\begin{array}{cccccccccccc}
1/d & 0 & \cdots & 0 & \alpha & 0 & \cdots & 0 & \beta & 0 & \cdots & 0 \\
0 & 1/d & 0 & \ddots & 0 & \alpha & \ddots & \ddots & 0 & \beta & \ddots & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & 1/d & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \beta \\
\alpha & 0 & \ddots & \ddots & 1/d & \ddots & \ddots & \ddots & \alpha & 0 & \ddots & 0 \\
0 & \alpha & \ddots & \ddots & \ddots & 1/d & \ddots & \ddots & 0 & \alpha & \ddots & \vdots \\
\vdots & 0 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 0 \\
0 & \ddots & \ddots & \alpha & 0 & \ddots & \ddots & 1/d & \ddots & \ddots & \ddots & \alpha \\
\beta & 0 & \ddots & 0 & \alpha & \ddots & \ddots & \ddots & 1/d & \ddots & \ddots & 0 \\
0 & \beta & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & 1/d & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \beta & 0 & \cdots & 0 & \alpha & 0 & \cdots & 0 & 1/d
\end{array}\right]
\end{array}. \tag{3.1}
$$

It's important to note that in this layout, the indices i, as indicated by the column on the left and the row above the matrix, correspond to the $(i+1)$-th column and row of the matrix.

Theorem 3.2. *A $d \times d$ symmetric Toeplitz matrix with $1/d$ main diagonal elements and positive elements on the m- and $(m+n)$-diagonal, as specified in* (3.1)*, is in $\mathcal{B}_d$ if and only if the following conditions are satisfied:*

Case 1. *For $m = n$, the conditions vary as follows:*

For $5m+1 \le d$,	$\alpha, \beta \ge 0;\ \alpha + 4\beta \le 2/d;\ 2\alpha - \beta \le 1/d;$
For $4m+1 \le d < 5m+1$,	$\alpha \ge 0;\ 0 \le \beta \le 1/(2d);\ \alpha + \beta \le 1/d;\ 2\alpha - \beta \le 1/d;$
For $3m+1 \le d < 4m+1$,	$\alpha, \beta \ge 0;\ \alpha + \beta \le 1/d;\ 2\alpha - \beta \le 1/d;$
For $2m+1 \le d < 3m+1$,	$\alpha \ge 0;\ 0 \le \beta \le 1/d;\ 2\alpha - \beta \le 1/d.$

Case 2. *For $m \ne n$, the conditions vary as follows:*

For $2m+2n+1 \le d$,	$\alpha, \beta \ge 0; \quad 2\alpha + 2\beta \le 1/d;$
For $2m+n+1 \le d < 2m+2n+1$,	$\alpha, \beta \ge 0; \quad 2\alpha + \beta \le 1/d;$
For $m+n+1 \le d < 2m+n+1$,	$\alpha, \beta \ge 0; \quad \alpha + \beta \le 1/d.$

Proof. For Case 1, when $m = n$, by permuting the rows and columns of (3.1), we can obtain the following block diagonal matrix $\mathbf{B}$, where each block is a two-dependent matrix:

$$\mathbf{B} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_m \end{bmatrix}$$

where

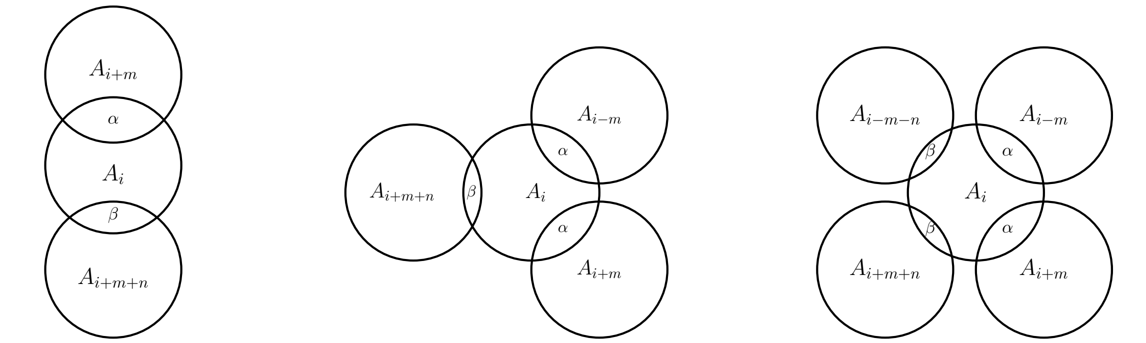
$$\mathbf{A}_i = \begin{bmatrix} 1/d & \alpha & \beta & 0 & \cdots & 0 \\ \alpha & 1/d & \alpha & \ddots & \ddots & \vdots \\ \beta & \alpha & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \alpha & \beta \\ \vdots & \ddots & \ddots & \alpha & 1/d & \alpha \\ 0 & \cdots & 0 & \beta & \alpha & 1/d \end{bmatrix}, \tag{3.2}$$

is a $k_i \times k_i$ matrix incorporating the $i,\ i+m, \ldots, i+(k_i-1)m$-th rows and columns of (3.1) with $k_i = \lceil (d-i+1)/m \rceil$, for $i = 1, \ldots, m$. Note that a matrix of the form (3.1) is in $\mathcal{B}_d$ if and only if $\mathbf{B}$ is in $\mathcal{B}_d$ (see Lemma 12 in [17]).

Now, let's consider the subcase $5m+1 \le d$ (or equivalently, $k_1 \ge 6$). For the only if part, if $\mathbf{B}$ is in $\mathcal{B}_d$, then $\mathbf{A}_i$ must be in $\mathcal{B}_{k_i}$, for $i = 1, \ldots, m$, respectively. The necessary and sufficient condition for $\mathbf{A}_1$ to be in $\mathcal{B}_{k_1}$ is given in Proposition 21 of [17], which is $\alpha, \beta \ge 0;\ \alpha + 4\beta \le 2/d;\ 2\alpha - \beta \le 1/d$. Note that $\mathbf{A}_i$, for $i = 2, \ldots, m$, either equals $\mathbf{A}_1$ or has one dimension less, then the conditions ensuring that $\mathbf{A}_i$ is in $\mathcal{B}_{k_i}$, for $i = 2, \ldots, m$ is identical or less stringent to the previous conditions (see Remark 22 of [17]). This completes the proof for this subcase. For the if part, given α and β satisfy these conditions, then $\mathbf{A}_1$ is in $\mathcal{B}_{k_1}$. If $k_i \ge 6$, the conditions for α and β remain the same, implying that $\mathbf{A}_i$ is in $\mathcal{B}_{k_i}$. If $k_i = 5$, the constraints on α and β are less restrictive than for $k_i = 6$ (see Remark 22 of [17]), confirming $\mathbf{A}_i$ is in $\mathcal{B}_{k_i}$. Therefore, $\mathbf{A}_i$ is in $\mathbf{B}_{k_i}$, for $i = 1, \ldots, m$, which implies that $\mathbf{B} \in \mathcal{B}_d$.

The following three subcases of Case 1 are equivalent to that of $k_1 = 5$, 4, and 3, respectively. Using a similar argument as above, we can conclude that $\mathbf{B}$ in $\mathcal{B}_d$ if and only if $\mathbf{A}_1$ in $\mathcal{B}_{k_1}$. Again, the necessary and sufficient conditions for $\mathbf{A}_1$ in $\mathcal{B}_{k_1}$ are given in Remark 22 of [17]. We observe that $2m+1 \ge d$, and this completes the proof of Case 1.

For Case 2, where $m \ne n$, unlike case 1, there are no 3×3 principal submatrices of $\mathbf{B}$ where all of its off-diagonal elements are positive. We consider the first subcase, where $2m+2n+1 \ge d$. To prove the only if part, suppose $\mathbf{B}$ is in $\mathcal{B}_d$. Let the sets A_i, for $i = 1, \ldots, d$, be those corresponding to $\mathbf{B}$ as specified in (3.1).



(A) $i = 1, \dots, m$ case (B) $i = m+1, \dots, m+n$ case (C) $i = m+n+1, \dots, d-m-n$ case

FIGURE 1. Venn Diagram for Sets A_i, $i = 1, \dots, d-m-n$

It is easy to see that A_i, for $i = 1, \dots, d$, intersect with two, three, or four sets according to the following five patterns:

$$
\begin{aligned}
&\Pr(A_i \cap A_{i+m}) = \alpha \text{ and } \Pr(A_i \cap A_{i+m+n}) = \beta, \text{ for } i = 1, \dots, m; \\
&\Pr(A_i \cap A_{i-m}) = \Pr(A_i \cap A_{i+m}) = \alpha \text{ and} \\
&\Pr(A_i \cap A_{i+m+n}) = \beta, && \text{for } i = m+1, \dots, m+n; \\
&\Pr(A_i \cap A_{i-m}) = \Pr(A_i \cap A_{i+m}) = \alpha \text{ and} \\
&\Pr(A_i \cap A_{i-m-n}) = \Pr(A_i \cap A_{i+m+n}) = \beta, && \text{for } i = m+n+1, \dots, d-m-n; \\
&\Pr(A_i \cap A_{i-m}) = \Pr(A_i \cap A_{i+m}) = \alpha \text{ and} \\
&\Pr(A_i \cap A_{i-m-n}) = \beta, && \text{for } i = d-m-n+1, \dots, d-m; \\
&\Pr(A_i \cap A_{i-m}) = \alpha \text{ and } \Pr(A_i \cap A_{i-m-n}) = \beta, \text{ for } i = d-m+1, \dots, d.
\end{aligned}
$$

Note that the last two patterns are similar to that of the first two patterns, respectively, so we can only focus on the first three. The Venn diagrams illustrating the sets A_i, for $i = 1, \dots, d-m-n$, are shown in Figure 1. By imposing the non-negativity constraint on the probability of each set and the constraint that $\Pr(A_i) = 1/d$, for $i = 1, \dots, d$, we obtain the following inequalities:

$$\alpha, \beta \geq 0; \tag{3.3}$$

$$\alpha + \beta \leq 1/d; \tag{3.4}$$

$$2\alpha + \beta \leq 1/d; \tag{3.5}$$

$$2\alpha + 2\beta \leq 1/d. \tag{3.6}$$

These constraints become more restrictive from (3.4) to (3.6). This completes the proof for the only if part in the first subcase. Similar arguments can be made for the second subcase $2m+n+1 \leq d < 2m+2n+1$ and the third subcase $m+n+1 \leq d < 2m+n+1$. In the second subcase, the constraints remain (3.3)-(3.5), and in the third subcase, the constraints remain (3.3) and (3.4). Observing that $d \geq m+n+1$, these completes the proof for the only if part in all subcases.

For the if part, let us initially focus on the first subcase. We will show that the matrix $\mathbf{B} = (b_{ij})$ takes the form of (3.1), for α, β satisfying $2\alpha + 2\beta \leq 1/d$, belongs to $\mathcal{B}_d$, for any $2m+2n+1 \leq d$. To validate this, we need to establish the existence of events A_i, $i = 1, \dots d$, on a probability space with identical probabilities of $1/d$ such that $b_{ij} = \Pr(A_i \cap A_j)$, for $1 \leq i, j \leq d$. To this end, we construct disjoint sets on a non-atomic probability space. These sets include C_i, for $i = 1, \dots d-m$, D_i, for $i = 1, \dots d-m-n$, E_i, for $i = 1, \dots m, d-m+1, \dots, d$, F_i, for $i = m+1, \dots m+n, d-m-n+1, \dots, d-m$, and G_i, for $i = m+n+1, \dots, d-m-n$, with probabilities

$$
\begin{aligned}
&\Pr(C_i) = \alpha; \quad \Pr(D_i) = \beta; \quad \Pr(E_i) = 1/d - \alpha - \beta; \\
&\Pr(F_i) = 1/d - 2\alpha - \beta; \quad \text{and} \quad \Pr(G_i) = 1/d - 2\alpha - 2\beta.
\end{aligned}
$$

This can be done as the sum of the probabilities of all above-mentioned sets are

$$\begin{aligned}(d-m)\alpha + (d-m-n)\beta + 2m(1/d-\alpha-\beta)&\\ + 2n(1/d-2\alpha-\beta) + (d-2m-2n)(1/d-2\alpha-2\beta)&\\ = 1-(d-m)\alpha-(d-m-n)\beta < 1.&\end{aligned}$$

Define

$$A_i = \begin{cases} C_i \cup D_i \cup E_i, & \text{for } i = 1,\ldots,m;\\ C_{i-m} \cup C_i \cup D_i \cup F_i, & \text{for } i = m+1,\ldots,m+n;\\ C_{i-m} \cup C_i \cup D_{i-m-n} \cup D_i \cup G_i, & \text{for } i = m+n+1,\ldots,d-m-n;\\ C_{i-m} \cup C_i \cup D_{i-m-n} \cup F_i, & \text{for } i = d-m-n+1,\ldots,d-m;\\ C_{i-m} \cup D_{i-m-n} \cup E_i, & \text{for } i = d-m+1,\ldots,d.\end{cases}$$

We see that $\Pr(A_i \cap A_j) = b_{ij}$, for $1 \le i, j \le d$, which completes the proof for the first subcase. Similarly, we can construct sets that satisfy the other two cases. □

Example 3.1. *Consider two distinct* 13×13 *matrices,* $\mathbf{B}_1$ *and* $\mathbf{B}_2$*, defined as follows:*

$$\mathbf{B}_1 = \begin{bmatrix} 1/13 & 0 & 0 & \alpha & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1/13 & 0 & 0 & \alpha & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & 0 & \beta & 0 & 0 & 0 & 0\\ \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & 0 & \beta & 0 & 0 & 0\\ 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & 0 & \beta & 0 & 0\\ 0 & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & 0 & \beta & 0\\ \beta & 0 & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & 0 & \beta\\ 0 & \beta & 0 & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & 0\\ 0 & 0 & \beta & 0 & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & \beta & 0 & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha\\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & \alpha & 0 & 0 & 1/13 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & \alpha & 0 & 0 & 1/13 \end{bmatrix},$$

where $m = n = 3$ *aligns with the second subcase of Case 1.*

$$\mathbf{B}_2 = \begin{bmatrix} 1/13 & 0 & 0 & \alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1/13 & 0 & 0 & \alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0\\ \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & \beta & 0 & 0 & 0 & 0\\ 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & \beta & 0 & 0 & 0\\ \beta & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & \beta & 0 & 0\\ 0 & \beta & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & \beta & 0\\ 0 & 0 & \beta & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0 & \beta\\ 0 & 0 & 0 & \beta & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & 0 & \beta & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0 & \alpha\\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & \alpha & 0 & 0 & 1/13 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 & \alpha & 0 & 0 & 1/13 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 & \alpha & 0 & 0 & 1/13 \end{bmatrix},$$

where $m = 3$ *and* $n = 2$*, corresponding to the first subcase of Case 2. By Theorem 3.2,* $\mathbf{B}_1$ *is a BCM if and only if* $\alpha \ge 0$; $0 \le \beta \le 1/26$; $\alpha + \beta \le 1/13$; $2\alpha - \beta \le 1/13$*, while* $\mathbf{B}_2$ *is a BCM if and only if* $\alpha, \beta \ge 0$; $2\alpha + 2\beta \le 1/13$.

Remark 3.3. *In the special case where* $m = n = 1$*, the matrix in* (3.1) *is a two-dependent matrix, meaning only the first two diagonals are positive. The conditions for this case align with those stated in Proposition 21 and Remark 22 of* [17]*, underpinning Case 1.*

To broaden the application of the specific case outlined in (3.1) towards the complete positivity of general symmetric Toeplitz matrices featuring two non-zero coefficients, we adapt the conditions presented in Theorem 3.2. This adaptation involves the substitution of α with $\beta/(d\alpha)$ and β with $\gamma/(d\alpha)$. By implementing these substitutions, we formulate the following corollary, thereby expanding our analysis to encompass a more extensive array of symmetric Toeplitz matrices:

Corollary 3.4. *A $d \times d$ symmetric Toeplitz matrix with α main diagonal elements and positive elements β and γ only on the m- and $(m+n)$-diagonal of the form*

$$\begin{array}{c} \\ \\ 1 \\ \vdots \\ \\ m \\ \\ \vdots \\ \\ m+n \\ \\ \vdots \\ d-1 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} & 1 & \dots & & m & & \dots & & m+n & & \cdots & d-1 \end{array} \\
\begin{bmatrix}
\alpha & 0 & \cdots & 0 & \beta & 0 & \cdots & 0 & \gamma & 0 & \cdots & 0 \\
0 & \alpha & 0 & \ddots & 0 & \beta & \ddots & \ddots & 0 & \gamma & \ddots & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \alpha & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \gamma \\
\beta & 0 & \ddots & \ddots & \alpha & \ddots & \ddots & \ddots & \beta & 0 & \ddots & 0 \\
0 & \beta & \ddots & \ddots & \ddots & \alpha & \ddots & \ddots & 0 & \beta & \ddots & \vdots \\
\vdots & 0 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 0 \\
0 & \ddots & \ddots & \beta & 0 & \ddots & \ddots & \alpha & \ddots & \ddots & \ddots & \beta \\
\gamma & 0 & \ddots & 0 & \beta & \ddots & \ddots & \ddots & \alpha & \ddots & \ddots & 0 \\
0 & \gamma & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \alpha & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \gamma & 0 & \cdots & 0 & \beta & 0 & \cdots & 0 & \alpha
\end{bmatrix}
\end{array}.$$

is completely positive if the following conditions are satisfied:

Case 1. *For $m = n$, the conditions vary as follows:*

$$\begin{array}{lr}
\textit{For } 5m+1 \le d, & \alpha,\beta,\gamma \ge 0;\ \beta + 4\gamma \le 2\alpha;\ 2\beta - \gamma \le \alpha; \\
\textit{For } 4m+1 \le d < 5m+1, & \beta \ge 0;\ 0 \le \gamma \le \alpha/2;\ \beta + \gamma \le \alpha;\ 2\beta - \gamma \le \alpha; \\
\textit{For } 3m+1 \le d < 4m+1, & \beta,\gamma \ge 0;\ \beta + \gamma \le \alpha;\ 2\beta - \gamma \le \alpha; \\
\textit{For } 2m+1 \le d < 3m+1, & \beta \ge 0;\ 0 \le \gamma \le \alpha;\ 2\beta - \gamma \le \alpha.
\end{array}$$

Case 2. *For $m \neq n$, the conditions vary as follows:*

$$\begin{array}{lr}
\textit{For } 2m+2n+1 \le d, & \alpha,\beta,\gamma \ge 0; \quad 2\beta + 2\gamma \le \alpha; \\
\textit{For } 2m+n+1 \le d < 2m+2n+1, & \alpha,\beta,\gamma \ge 0; \quad 2\beta + \gamma \le \alpha; \\
\textit{For } m+n+1 \le d < 2m+n+1, & \alpha,\beta,\gamma \ge 0; \quad \beta + \gamma \le \alpha.
\end{array}$$

Remark 3.5. *It is well-established that any symmetric, nonnegative, diagonally dominant matrix is completely positive. That is, for a matrix $\mathbf{A}_{d\times d} = (a_{ij})$, where $a_{ij} \ge 0$, $a_{ij} = a_{ji}$, and*

$$a_{ii} \ge \sum_{k\neq i} a_{ik}, \quad \textit{for all } i,$$

then $\mathbf{A}$ belongs to $\mathcal{C}_d$. However, the matrices discussed in Corollary 3.4 are not necessarily diagonally dominant. For instance, consider the completely positive matrix $\mathbf{J}_{3\times 3}$ with $m = n = 1$, which is not diagonally dominant.

4. Discussion and Conclusion

Recall that the matrix in (1.1) can be expressed as $T_d(f)$, where the generating function is given by

$$f(\theta) := \alpha_0 + 2\sum_{i=1}^{d-1} \alpha_i \cos(i\theta), \quad \theta \in [0, \pi],$$

see [6] and [14] and the references therein. It is known that if f is nonnegative on $[0, \pi]$ and not almost surely equal to 0, then the Toeplitz matrix $T_d(f)$ is positive definite for any positive order d (see Corollary 5.1 of [14]). Moreover, its eigenvalues can be computed using highly efficient algorithms, even when the coefficients α_i are complex, see [4] and [12].

When considering the matrix in (3.1), its generating function specializes to the trigonometric polynomial

$$1/d + 2\alpha \cos(m\theta) + 2\beta \cos((m + n)\theta).$$

Ensuring that this polynomial remains nonnegative over $[0, \pi]$ guarantees the matrix is positive definite, making it a valuable preliminary check for complete positivity.

This paper primarily examines symmetric Toeplitz matrices and identifies the key conditions under which they are Bernoulli compatible matrices—a classification of particular interest in extreme value theory. We further extend our analysis to the concept of complete positivity, focusing on symmetric Toeplitz matrices in which only two of the α_i coefficients are nonzero. For these matrices, we establish sufficient conditions that ensure complete positivity. These findings not only improve the computational feasibility of working with such matrices but also open new research directions in the broader domain of Toeplitz matrices and their applications.

5. Conflict of Interest

There is no conflict of interest.

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