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A Sufficient Condition for a Certain Set of Positive Integers to be Subset-Sum-Distinct

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Abstract. Let S be a set of positive integers. We say that S is a subset-sum-distinct set (briefly, S is an SSD-set) if for any two finite subsets X, Y of S,

$$\sum_{x\in X} x = \sum_{y\in Y} y \quad \Rightarrow \quad X = Y.$$

For fixed positive integers n and p, we set

$$S(n,p) := \{1^p, 2^p, 3^p, \dots, n^p\}.$$

We prove a sufficient condition for which $S(n,p)$ is an SSD-set.

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1. Introduction

Let S be a set of positive integers. We say that S is a subset-sum-distinct set (briefly, S is an SSD-set) if for any two finite subsets X, Y of S,

$$\sum_{x\in X} x = \sum_{y\in Y} y \quad \Rightarrow \quad X = Y.$$

One of the most interesting and natural SSD-sets is

$$S := \{1, 2, 2^2, 2^3, \dots\}.$$

By the uniqueness of binary expansion, S is certainly an SSD-set.

Stimulated by Erdös' open question ([7, p. 114, problem C8]), finite dense SSD-sets have been considered by many mathematicians (see [1],[2],[3],[4],[5, pp. 59–60],[6]).

On the other hand, it is one of the hardest problems in computer science to determine whether a given set S is an SSD-set. The difficulty arises from the fact that we need to consider all subsets of S.

The purpose of this paper is as follows: For fixed positive integers n and p, we set

$$S(n,p) := \{1^p, 2^p, 3^p, \dots, n^p\}. \tag{1}$$

We prove a sufficient condition for which $S(n,p)$ is an SSD-set.

2. Main result

Concerning $S(n,p)$ in (1), we pose the following:

Problem 1.

(i) For a fixed n, find a condition on p for which $S(n,p)$ is an SSD-set.
(ii) As a special case of (i), is it true that if $S(n,r)$ is an SSD-set for some positive integer r, then is so $S(n,r+1)$?

2.1. **Motivation.** The motivation to consider Problem 1 is as follows. For a Morse function $f : M \to \mathbb{R}$ on a compact manifold M, we define the fiber product by

$$C(f) := \{(u,v) \in M \times M \mid f(u) = f(v)\}.$$

As explained in [8], it is worthwhile to obtain topological information on $C(f)$.

As f, we consider Morse functions on $U(n)$. Here $U(n)$ denotes the unitary group of degree n consisting of $n \times n$ unitary matrices.

Recall from [9] that correspondingly to a choice of real numbers

$$0 < c_1 < c_2 < \cdots < c_n,$$

we obtain the canonical Morse function on $U(n)$. When a positive integer p is fixed and c_i is defined by $c_i = i^p$ for $1 \le i \le n$, we write the Morse function by $f_{n,p}$.

We already know the following formula

$$\chi(C(f_{n,p})) = (-1)^n \int_0^1 \prod_{j=1}^{n} \left(4\sin^2(\pi j^p x)\right) dx, \tag{2}$$

where $\chi(C(f_{n,p}))$ denotes the Euler characteristic of the space $C(f_{n,p})$.

For special cases, we can simplify (2) by the following:

Lemma 2. *The equation*

$$\chi(C(f_{n,p})) = (-2)^n \tag{3}$$

holds if and only if $S(n,p)$ is an SSD-set. (Note that in this case, the right-hand side of (3) *does not depend on p.)*

Lemma 2 is the motivation for considering Problem 1.

2.2. **Example.** For a fixed n, we set

$$\lambda(n) := \min\{p \mid S(n,p) \text{ is an SSD-set}\}.$$

With the aid of a computer, we have the following Table 1.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lambda(n)$	1	1	2	2	3	4	4	4	5	5	5	6	6	6	6

16	17	18	19	20	21	22	23	24	25	26	27
7	7	7	7	8	8	9	9	9	9	9	9

TABLE 1. $\lambda(n)$ for $1 \leq n \leq 27$.

Remark 3. As Problem 1 (ii) indicates, it is not known whether, for example, $S(27,p)$ is an SSD-set for all $p > 9$.

2.3. **Main result.** Now we give an answer to Problem 1.

Theorem 4. *If $p \geq n$, then $S(n,p)$ is an SSD-set.*

3. Proof of Theorem 4

We prove Theorem 4 by induction on n.

Base case: Since $S(1,p) = \{1\}$, Theorem 4 holds for $n = 1$.

Induction step: We assume that $S(n-1,q)$ is an SSD-set for $q \geq n-1$. We need to prove that $S(n,p)$ is an SSD-set for $p \geq n$. Using the inductive hypothesis, it will suffice to find p which satisfies $p \geq n-1$ and the following Condition 5:

Condition 5 (Condition on p)**.** We require that p satisfies the following statement: Let X and Y be any subsets of $S(n,p)$ satisfying the following (i) and (ii):

(i) $X \cap Y = \varnothing$, and
(ii) $n^p \in Y$.

Then we require that

$$\sum_{x \in X} x \neq \sum_{y \in Y} y.$$

In order to find p which satisfies Condition 5, note that if p satisfies

$$\sum_{k=1}^{n-1} k^p < n^p, \tag{4}$$

then p satisfies Condition 5. We study which p satisfies (4). Considering the lower Riemann sum, we have

$$\sum_{k=1}^{n-1} k^p < \int_1^n x^p \, dx. \tag{5}$$

We consider the following inequality:

$$\int_1^n x^p \, dx < n^p \tag{6}$$

Thanks to (5), if p satisfies (6), then p satisfies (4).

We shall prove that if $p \geq n$, then (6) holds. In fact,

$$\int_1^n x^p \, dx = \frac{n^{p+1} - 1}{p+1} < \frac{n^{p+1}}{p} \leq \frac{n^{p+1}}{n} = n^p.$$

Now we have proved that when $p \geq n$, (6) holds. Hence (4) also holds. Consequently, Theorem 4 holds.

4. Conclusions

We show that there are two reasons why our bound $p \geq n$ in Theorem 4 is far from being the sharp bound. To study concretely, we consider the case $n = 20$. By Table 1, $S(20, p)$ is an SSD-set when $p = 8$.

The first reason. In order to avoid the difficulty of considering all subsets of $S(20, p)$, we replace the SSD-condition by (4). More precisely, we have shown that if p satisfies

$$\sum_{k=1}^{19} k^p < 20^p, \tag{7}$$

then $S(20, p)$ is an SSD-set. Note that (7) is only a sufficient condition, but far from being the necessary condition for $S(20, p)$ to be an SSD-set.

The second reason. Direct computations show that (7) holds for

$$p \geq 13. \tag{8}$$

But since the computations are troublesome, we proceeded the argument that if p satisfies

$$\int_1^{20} x^p \, dx < 20^p, \tag{9}$$

then p satisfies (7). We obtain from (9) that (7) holds for

$$p \geq 20. \tag{10}$$

Note that (8) and (10) have difference.

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6. Conflict of Interest

There is no conflict of interest.

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