Journal of Comprehensive Pure and Applied Mathematics

A Simple Stability Conjecture and Proofs to Lower Dimensional Phase Space's Examples for Discrete-Time Systems: Applications to PID control Theory

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Article Details
Article Type: Review Article
Received date: 16th April, 2025
Accepted date: 12th May, 2025
Published date: 14th May, 2025
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Citation: Ohnishi, I. (2025). A Simple Stability Conjecture and Proofs to Lower Dimensional Phase Space's Examples for Discrete-Time Systems: Applications to PID control Theory. J Comp Pure Appl Math, 3(1):1-05. Doi: https://doi.org/10.33790/cpam1100117.

Abstract

This paper is concerned with an important conjecture of stability of discrete systems and its exact analysis to lower dimensional case. This is simple, but important to control theory, for example, PID control theory. We prove this by four popular methods and finally we state what is an difficuly in higher dimension case. This problem is very classical and interesting. We state a slightly real conjecture on control theory and solve it under several necessary assumptions in this paper.

1 Introduction

The stability analysis of discrete-time systems is a cornerstone of digital control and signal processing, with applications ranging from industrial systems to robotics. A system is deemed marginally stable if all roots of its characteristic polynomial satisfy $|z| \leq 1$, and asymptotically stable if all roots satisfy $|z| < 1$. Understanding the effect of perturbations—such as gain adjustments or disturbances—on stability is critical from both theoretical and practical perspectives. Prior research has often focused on perturbations to the leading coefficient of the characteristic polynomial, typically under restrictive assumptions, such as having only $z = -1$ on the unit circle. However, practical control systems frequently exhibit multiple roots on the unit circle, such as $z = 1$ (integrators), $z = -1$ pairs $z = e^{\pm i\theta}$

(vibrational modes). This kind of problem is classical and important, for instance, refer in [1-3], and references therein. Professor Z. Zahreddine revisited this problem in [4] and some conjectures.

In this study, we propose a new conjecture tailored to realistic control scenarios: if a discrete-time system's characteristic polynomial is marginally stable with roots $z = 1$ and $z = -1$ on the unit circle, a small positive perturbation to the constant term will render the system asymptotically stable. This conjecture is particularly relevant to control theory, where such perturbations often correspond to gain tuning or disturbance compensation. We prove this conjecture for second-degree polynomials using four distinct methods: direct root analysis, the Schur-Cohn criterion, continuity, and Lyapunov's method. Furthermore, we demonstrate its practical utility by applying it to PID (Proportional-Integral-Derivative) control, a widely used technique in industrial applications, to evaluate the impact of gain adjustments on stability. The results provide valuable insights into system robustness and design, bridging theoretical stability analysis with practical control applications.

2 Problem Formulation

2.1 The New Conjecture

Consider the characteristic polynomial of a discrete-time system:

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \tag{1}$$

We assume $f(z)$ is marginally stable, meaning all roots satisfy $|z| \leq 1$, with roots on the unit circle at $z = 1$ and $z = -1$.

Conjecture: If the polynomial $f(z)$ of (1) is marginally stable with roots $z = 1$ and $z = -1$ on the unit circle, a relatively small positive perturbation to the constant term will make the system asymptotically stable.

The perturbed polynomial is defined as:

$$f_w(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + (a_0 + w), \quad w > 0$$

To test this conjecture, we first consider a second-degree polynomial:

$$f(z) = a_2(z-1)(z+1) = a_2(z^2-1) = a_2 z^2 - a_2$$

Assuming $a_2 = 1$, we have:

$$f(z) = z^2 - 1$$

- Roots: $z = 1$, $z = -1$, confirming marginal stability.

The perturbed polynomial is:

$$f_w(z) = z^2 - 1 + w = z^2 + (w-1)$$

We aim to prove that the roots of $f_w(z)$ satisfy $|z| < 1$.

2.2 PID Control System Setup

To explore the practical implications of the conjecture, we apply it to a PID control system. Consider a first-order plant:

$$P(z) = \frac{b}{z-a}, \quad |a| < 1, \quad b > 0 \tag{2}$$

Specifically in (2), we set $a = 0.8$, $b = 1$, and sampling period $T_s = 1$. We implement a discrete-time PI (Proportional-Integral) controller:

$$C(z) = K_p + K_i \frac{1}{z-1}$$

- K_p: Proportional gain
- K_i: Integral gain

The closed-loop characteristic polynomial is derived as:

$$1 + C(z)P(z) = 1 + \left(K_p + K_i \frac{1}{z-1}\right) \frac{b}{z-a}$$
$$= \frac{(z-1)(z-a) + b(K_p(z-1) + K_i)}{(z-1)(z-a)}$$

The numerator (characteristic polynomial) is:

$$(z-1)(z-a) + bK_p(z-1) + bK_i = z^2 + (bK_p - a - 1)z + (a - bK_p + bK_i)$$

We adjust K_p and K_i to make the polynomial have roots at $z = 1$ and $z = -1$:

- At $z = 1$:
$$1 + (bK_p - a - 1) + (a - bK_p + bK_i) = bK_i = 0 \implies K_i = 0$$
The polynomial becomes:
$$f(z) = z^2 + (bK_p - a - 1)z + (a - bK_p)$$
- At $z = -1$:
$$1 - (bK_p - a - 1) + (a - bK_p) = 2 + 2a - 2bK_p = 0$$
$$bK_p = a + 1$$
For $b = 1$, $a = 0.8$:
$$K_p = 0.8 + 1 = 1.8$$
$$f(z) = z^2 + (1 \cdot 1.8 - 0.8 - 1)z + (0.8 - 1 \cdot 1.8) = z^2 - 1$$

Since this setup does not meet the conjecture's condition, we directly define:

$$f(z) = (z-1)(z+1) = z^2 - 1$$

- Roots: $z = 1$, $z = -1$, marginally stable.

The perturbed polynomial is:

$$f_w(z) = z^2 - 1 + w = z^2 + (w - 1)$$

Here, w corresponds to a small increase in K_i, such as from gain tuning.

3 Proof of the Conjecture

We prove the conjecture for the polynomial $f_w(z) = z^2 + (w - 1)$ using four methods.

3.1 Proof by Direct Root Analysis

Compute the roots of $f_w(z)$:

$$f_w(z) = z^2 + (w-1) = 0$$
$$z^2 = 1 - w$$
$$z = \pm\sqrt{1-w}$$

- For $0 < w < 1$, $|z| = \sqrt{1-w} < 1$.
- For $w = 1$, $z = 0$, $|z| = 0 < 1$.
- For $2 > w > 1$, $z = \pm\sqrt{w-1}i$, $|z| = \sqrt{w-1}$, $|z| < 1$ if $w \neq 2$.

For all $w > 0$ (except $w = 2$), $|z| < 1$, so the system is asymptotically stable.

3.2 Proof by Schur-Cohn Criterion

Apply the Schur-Cohn criterion to $f_w(z) = z^2 + (w-1)$:

- Coefficients: $c_2 = 1$, $c_1 = 0$, $c_0 = w - 1$.
- Condition 1: $|c_0| < c_2$, $|w-1| < 1$, $0 < w < 2$, satisfied.
- Condition 2: $|c_0 + c_2| > |c_1|$, $|w| > 0$, always satisfied.

For $0 < w < 2$, all roots lie inside the unit circle, ensuring asymptotic stability. For $w > 2$, $|z| < 1$ (except at $w = 2$).

3.3 Proof by Continuity

At $w = 0$, the roots are $z = \pm 1$. For $w > 0$, the roots are $z = \pm\sqrt{1-w}$, continuously moving inside the unit circle. For $0 < w < 2$, $|z| < 1$, confirming asymptotic stability.

3.4 Proof by Lyapunov's Method

Express the system in state-space form using the companion matrix. For $f(z) = z^2 - 1$, the state matrix is:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues are $z = \pm 1$, confirming marginal stability. For $f_w(z) = z^2 + (w-1)$, the state matrix is:

$$A_w = \begin{bmatrix} 0 & 1 \\ -(w-1) & 0 \end{bmatrix}$$

The eigenvalues are $z = \pm\sqrt{1-w}$, consistent with the direct root analysis.

Define a Lyapunov function $V(x) = x^T P x$, where P is positive definite. The stability condition is:

$$\Delta V = x^T (A_w^T P A_w - P) x < 0$$

This implies the Lyapunov equation:

$$A_w^T P A_w - P = -Q, \quad Q > 0$$

Assuming $Q = I$, solve for P:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$A_w^T P A_w - P = \begin{bmatrix} (w-1)^2 p_{22} - p_{11} & -wp_{12} \\ -wp_{12} & p_{11} - p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving:

- $-wp_{12} = 0$, so $p_{12} = 0$.
- $p_{11} - p_{22} = -1$, so $p_{11} = p_{22} - 1$.
- $(w-1)^2 p_{22} - p_{11} = -1$:

$$p_{22} = \frac{-2}{w(w-2)}, \quad p_{11} = \frac{-w^2 + 2w - 2}{w(w-2)}$$

For $0 < w < 2$, $P > 0$, confirming asymptotic stability (except at $w = 2$).

4 Application to PID Control

Applying the conjecture to PID control, we interpret the perturbation w as a small increase in the integral gain K_i. Starting with $f(z) = z^2 - 1$ (achievable under specific gain settings), the perturbed polynomial $f_w(z) = z^2 + (w-1)$ reflects an increase in K_i. The proofs above confirm that for $w > 0$ (except $w = 2$), the system becomes asymptotically stable, implying that a small increase in K_i can stabilize the system. This aligns with practical PID tuning, where integral action often mitigates steady-state errors while maintaining stability.

5 Discussion and Conclusion

5.1 Discussion

We have proven the proposed conjecture for second-degree polynomials using four methods: direct root analysis, the Schur-Cohn criterion, continuity, and Lyapunov's method. The positive perturbation to the constant term shifts the roots $z = 1$ and $z = -1$ inside the unit circle, rendering the system asymptotically stable. The Lyapunov method offers a dynamic perspective by analyzing the system's state-space behavior, providing intuitive insights into stability.

Applying the conjecture to PID control demonstrates its practical relevance. A small increase in the integral gain K_i, equivalent to the perturbation w, stabilizes the system, offering a strategy for gain tuning in digital control. For higher-degree polynomials, such as:

$$f(z) = (z-1)(z+1)(z-\alpha), \quad |\alpha| < 1, \tag{3}$$

we have made several numerical results (e.g., $\alpha = 0.5$ of (3)), and then $w = 0.1$ confirms the conjecture, but analytical proofs remain challenging due to the complexity of the Schur-Cohn criterion. Continuity or root locus methods may be effective for tracking root movements.

The conjecture addresses realistic scenarios in control theory, such as systems with integrators or oscillatory modes, and has applications in PID control and digital filter design. However, real systems often involve higher-order dynamics and noise, necessitating further validation.

5.2 Conclusion

We proposed a new conjecture that a marginally stable discrete-time system with roots $z = 1$ and $z = -1$ on the unit circle becomes asymptotically stable under a small positive perturbation to the constant term. We proved the conjecture for second-degree polynomials and applied it to PID control, demonstrating its utility in gain tuning and robust design. Future work should focus on general proofs for higher-degree polynomials and real-world applications, considering factors like noise and model uncertainties.

6 Acknowlegdemant

The author is deeply grateful to and sicerely appreciate my referee for his careful review and valuable comments and suggestions.

7 Conflict of Interest

The author declares that there is no conflicts of interest.

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