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Numerical solution to inverse scattering problem

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Abstract

A new method is given for the solution to inverse scattering problem. This method is based on the estimates of the solution to the Schrödinger equation. It is of interest to check the effectiveness of this method.

Key words: inverse scattering; numerical methods.

1 Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected smooth boundary S, $D' := \mathbb{R}^3 \setminus D$, $D \subset B_a := \{|x| \leq a\}$.

Let us assume that $q = q(x) \subset Q_a$, where $supp\, q \subset D$, $\|q\|^2 := \int_D |q|^2 dx < \infty$.

Consider the boundary value problem:

$$L_q u := (\nabla^2 + 1 - q(x))u = 0 \text{ in } D, \tag{1}$$

$$u = e^{i\theta \cdot s} \text{ on } S, \quad \theta \in M, \tag{2}$$

where $M := \{\theta : \theta \cdot \theta = 1\}$, $\theta \in \mathbb{C}^3$. Let us assume that all the Dirichlet eigenvalues of L_q in D are not equal to zero. If this is so, then the problem (1)-(2) is uniquely solvable for any $\theta \in \mathbb{C}^3$. One can check that for any $\xi \in \mathbb{R}^3$ one can find θ_1 and θ_2 in M such that

$$\theta_1 - \theta_2 = \xi, \quad |\theta_1| \to \infty, \ |\theta_2| \to \infty. \tag{3}$$

We leave for the reader to check (3) (a proof of (3) can be found in [1], p. 50, and in [3], p. 368).

Recall the known formula for the scattering amplitude:

$$A(\beta, \alpha) = -\frac{1}{4\pi} \int_D e^{-i\beta \cdot y} q(y) u(y, \alpha) dy, \tag{4}$$

where α and β are unit vectors in the directions of the incident and scattered waves and $u(y, \theta)$ is the scattering solution to (1). If $q \in Q_a$, then the scattering amplitude is analytic on M, and one can write

$$A(\theta_2, \theta_1) = -\frac{1}{4\pi} \int_D e^{-i\theta_2 \cdot y} q(y) u(y, \theta_1) dy, \tag{5}$$

where $\theta_1, \theta_2 \in M$.

Proposition 1. The knowledge of $A(\beta, \alpha)$ for all $\beta, \alpha \in S^2_1$ determines $q \in Q_a$ uniquely by the formula:

$$\tilde{q}(\xi) = \lim_{\theta_1 - \theta_2 = \xi, \, |\theta_j| \to \infty, \, j=1,2} A(\theta_2, \theta_1), \tag{6}$$

where $\tilde{q} := \int_D e^{i\xi \cdot x} q(x) dx$.

This result is proved in [2], pp. 264-267, see also [3], pp. 383-391.

In [2], p. 260, the special solutions to (1) in $\mathbb{R}^3$ was described:

$$\psi(x, \theta) = e^{i\theta \cdot x}(1 + R(x, \theta)), \quad \|R\| \le \frac{c}{\theta}, \quad |R| \le \Big(\frac{\log|\theta|}{|\theta|}\Big)^{1/2} = o(1), \tag{7}$$

where $\theta \in M$, $|\theta| \to \infty$.

If one replaces in (5) the scattering solution u to problem (1) in $\mathbb{R}^3$ by the special solution (7), neglect the term $R(x, \theta)$ and passes to the limit as in (6), then formula (6) will be obtained.

The difficulty is: we do not know the potential q, so we do not know the scattering solution and it is not clear how does one calculate the analytic continuation of the scattering amplitude.

The main results of this paper is the following Theorems.

Theorem 1. A solution in D to problem (1)-(2), which satisfies formula (7) with $R(x,\theta) = o(1)$ as $|\theta| \to \infty$, can be calculated numerically by solving problem (1)-(2) with $\theta \in M$, $|\theta| \to \infty$.

Let ϕ solve the problem

$$(\Delta + 1)\phi = 0 \text{ in } D, \quad \phi|_S = e^{i\theta\cdot s}.$$

Then $\phi = e^{i\theta\cdot x}$ in D, because $\theta \cdot \theta = 1$.

Theorem 2. One has

$$\lim_{|\theta|\to\infty} \frac{u}{\phi} = 1, \tag{8}$$

where u solves problem (1)–(2).

In Section 2 proofs are given.

2 Proofs

Proof of Theorem 1.

Because of our assumption that the Dirichlet operator L_q does not have zero eigenvalue in D, problem (1)-(2) has a solution for every $\theta \in M$, and this solution is unique. Let us prove that this solution is of the form (7). Let $G(x,y)$ be the Green's function:

$$L_q G = -\delta(x-y) \quad \text{in } D, \quad G|_S = 0. \tag{9}$$

This G exists, is unique, and

$$|G(x,y)| \leq \frac{c}{|x-y|}, \quad \sup_{x\in D} \int_D |\frac{\partial G(x,s)}{\partial N_s}| ds \leq c, \tag{10}$$

where $c > 0$ denotes various constants.

The unique solution to (1)-(2) is:

$$u(x,\theta) = -\int_D e^{i\theta\cdot s} \frac{\partial G(x,s)}{\partial N_s} ds, \tag{11}$$

where N_s is the outer normal to S at the points s. This formula is the Green's formula. Similarly,

$$\psi(x,\theta) = -\int_D \psi(s,\theta) \frac{\partial G(x,s)}{\partial N_s} ds, \tag{12}$$

where $e^{i\theta\cdot s} = u|_S$.

Denote $w := u - \psi$. Subtract equation (12) from equation (11) and get:

$$w(x,\theta) = -\int_D w(s,\theta)\frac{\partial G(x,s)}{\partial N_s}ds, \tag{13}$$

so

$$|w(x,\theta)| \leq \int_D |\frac{\partial G(x,s)}{\partial N_s}|ds \sup_{s\in S}|w(s,\theta)| \leq c\sup_{s\in S}|w(s,\theta)|. \tag{14}$$

Since $\frac{u}{\psi} = 1 + \frac{w}{\psi}$, one gets

$$u(s,\theta) = \psi(1 + \frac{w(s,\theta)}{\psi(s,\theta)}). \tag{15}$$

One has $u|_S = e^{i\theta\cdot s}$. Therefore $w(s,\theta) = u - \phi = R(s,\theta)e^{i\theta\cdot s}$, where $|R| = o(1)$ as $|\theta| \to \infty$. Thus, $|\frac{w(s,\theta)}{\psi(s,\theta)}| \leq o(1)$ and $u(x,\theta) = e^{i\theta\cdot x}(1 + o(1))$ as $|\theta| \to \infty$, $\theta \in M$.

Theorem 1 is proved. $\Box$

The u still depends on$q$. To avoid the difficulty, mentioned in the Introduction, we use Theorem 2.

Proof of Theorem 2.

Since $u(x,\theta) = e^{i\theta\cdot x}(1+o(1))$ as $|\theta| \to \infty$, $\theta \in M$ and $\phi = e^{i\theta\cdot x}$, it follows that relation (8) holds.

Theorem 2 is proved. $\Box$

Remark 1. Analytic continuation with respect to β and α from S^2 to M can be done using series in the spherical harmonics for the scattering amplitude (see [2], p.262 and [3], p.387-391).

Remark 2. A known elliptic estimate for the solution to (1)-(2) is $\|u\|_{H^1(D)} \leq c\|u\|_{H^{1/2}(S)}$, where $H^\alpha(D)$ is the Sobolev space. Therefore, $\|w\|_{H^1(D)} \leq c\|w\|_{H^{1/2}(S)} = c\|e^{i\theta\cdot s}R\|_{H^{1/2}(S)}$.

Numerical solution to problem (1)-(2) with any boundary data is done by any known simple and efficient method.

Demonstrating practical efficiency of the described method of solving the inverse scattering problem under the assumptions of this paper is an interesting open problem.

3 Conclusion

A numerical method is given for solving inverse potential scattering problem with compactly supported potential.

4 Disclosure statement.

There are no competing interests to declare. There is no financial support for this work.

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