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Open Questions For Deformations of Dotted Diagrams

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Abstract. A dotted diagram is a certain planar graph. The notion of a dotted diagram was introduced to treat a lattice diagram which is associated with a pair of partial matchings. We review the notions of a lattice diagram, and a dotted diagram and its deformations. Further we discuss several open questions related to dotted diagrams.

0.1. **Partial matchings and lattice diagrams.** The notion of a dotted diagram was introduced to treat a lattice diagram which is associated with a pair of partial matchings. In this section, we review partial matchings and lattice diagrams. See [4], but note that we revised terminologies in [5, Table 1]. We refer to [2] for basics of graph theory.

A *partial matching* is a finite graph with the vertex set $\{1, 2, \ldots, m\}$ for some positive integer m such that the vertices are of degree zero or one. We present a partial matching by

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the set of points in the xy-plane, called the *lattice presentation*, which consists of two pairs of lattice points (x, y) and (y, x) for each edge connecting the vertices x and y [4, Definition 3.1]. When we have a pair of partial matchings with the same set of vertices and the same number of edges, we obtain a *lattice diagram* P by the following method. We denote by Δ_0 and Δ_1 the lattice presentations of the pair of partial matchings in question. Then, in the xy-plane, put Δ_0 denoted by small black disks (called *initial vertices*) and Δ_1 by X marks (called *terminal vertices*). When an initial vertex v_0 does not coincide with a terminal vertex, there exists a unique pair of terminal vertices v_1 and v_1' such that v_0 and v_1, and v_0 and v_1' have the same x-coordinate and y-coordinate, respectively. So, connect v_0 and v_1, and v_0 and v_1' with edges, and equip them with orientations from v_0 and v_1, and v_1' and v_0, respectively. Then we obtain a graph in the xy plane which is "symmetric" with respect to the line $y = x$, which can be described by $P \cup (-P^*)$ for some graph P, where $-P^*$ is the orientation-reversed mirror image of P. We note that the choice of P is not unique. We call P a *lattice diagram* associated with Δ_0 and Δ_1 [4, Definition 4.3]. When we have a lattice diagram P as above, we have the notion of a *transformation of P along a rectangle*, which is associated with a certain transformation of a partial matching which switches parings of two edges. Further, we have the notion of a *dissolution* of P, and the *area* of a dissolution of P. See [4, Section 5] and [5, Section 2.2].

0.2. **Dotted diagrams.** Now, we review the notion of a dotted diagram [5]. A dotted diagram is a planar graph with vertices of degree 4 called *crossings* and vertices of degree 2 called *dots* [5, Definition 3.2]. A dotted diagram Γ can be regarded as an immersion of several oriented circles with transverse intersection points, and we assign each region (connected component of $\mathbb{R}^2 \backslash \Gamma$) the sum of rotation numbers. When we have a lattice diagram P, taking the "essential diagram" ∂P of P and deleting terminal vertices and regarding the result as a planar graph with crossings and dots (coming from initial vertices), we have a dotted diagram (see Figure 1). When a dotted diagram admits a lattice diagram, we call it an *admissible* dotted diagram. In [5, Definition 3.6], we defined deformations of dotted diagrams, using the notions of a *layer decomposition* and a *background* (see Figure 2). The deformations consist of 4 types, and they are well-defined up to a certain local move called a local move $\mathcal{E}$. We note that the deformations were introduced as those induced from transformations of lattice diagrams which realize dissolutions with minimal area. In [5, Definition 3.9], we defined a *reduced diagram* of a dotted diagram Γ, which is defined as a dotted diagram which is obtained from Γ by a finite sequence of deformations, and which cannot be applied a deformation any more. In [5], we showed several results including existence and uniqueness of a reduced diagram, which were each shown with some different assumptions, and we showed relation between the existence of a dissolution of a lattice diagram with minimal area, and having the empty diagram as a reduced diagram.

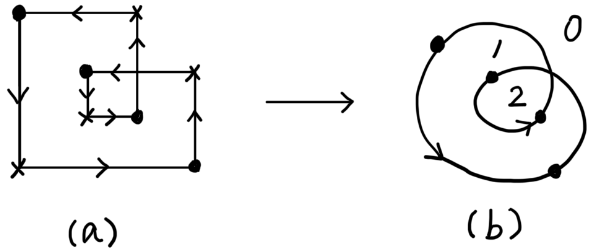

FIGURE 1. (a) A lattice diagram and (b) the associated dotted diagram. This figure is [5, Figure 2].

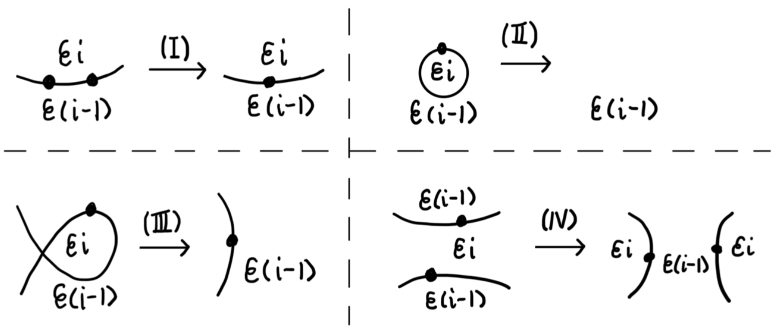

FIGURE 2. Deformations I–IV. For II, III and IV, the figures illustrate backgrounds; see [5, Definition 3.6] for details. This figure is [5, Figure 6].

0.3. **Open questions.** We list up open questions as follows.

Question 1. Construct examples of (both inadmissible and admissible) reduced diagrams as many as possible.

Question 2. Characterize or classify reduced diagrams.

In [6], we showed that an admissible dotted diagram is reducible if it is in a certain form consisting of several standard circles.

Question 3. Give a condition of "forms" of a dotted diagram to be a reduced diagram.

In [5, Theorem 3.11], we showed the existence of a "good" reduced diagram under deformations satisfying a certain condition.

Question 4. Give a simpler set of deformations which induce a unique reduced diagram for any given dotted diagram.

The deformations defined in [5, Definition 3.6] have a stronger condition than those which appear as transformations of lattice diagrams realizing the minimal area. More precisely, transformations of lattice diagrams realizing dissolutions with minimal area do not require deformations of dotted diagrams to satisfy the condition that each overlapping layer has the label ϵ. Indeed, we might be able to loosen the condition so that the deleting region has the total sum non-zero of its label and the labels of the overlapping layers. But then the deformations will not be well-defined.

Question 5. Give an improved set of deformations of dotted diagrams which are more fitting to transformations of lattice diagrams realizing dissolutions with minimal area.

Deformations of dotted diagrams seem to suggest relation to Reidemeister moves for knot diagrams [3]. Further, a sequence of deformations of a dotted diagram seem to suggest relation to a knotted surface in a 4-ball [1].

Question 6. Can we construct invariants of dotted diagrams using techniques of knot theory? Can we construct a surface in a 4-ball which can be studied using techniques of surface-knot theory?

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Competing interests

The author declares no competing interests.

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