Gexin Publications

Journal of Comprehensive Pure and Applied Mathematics

Fractional Calculus in Applied Mathematics: Historical Development, Modern Applications, and Future Perspectives

Chunyan Liu[1*], Yuxuan Yang[1], Yu Bai[1], Yan Zhang[1], Tami Yang[2], Maoxing Liu [1]

[1]School of Science, Beijing University of Civil Engineering and Architecture, Beijing 100044, China
[2]Business School, Beijing Information Science and Technology University, Beijing 102206, China

Article Details

Article Type: Commentary Article
Received date: 15th May, 2026
Accepted date: 18th September, 2026
Published date: 05th October, 2026
***Corresponding Author**: Chunyan Liu, School of Science, Beijing University of Civil Engineering and Architecture, Beijing 100044, China
Citation: Liu, C., Yang, Y., Bai, Y., Zhang, Y., Yang, T. and Liu, M. (2026). Fractional Calculus in Applied Mathematics: Historical Development, Modern Applications, and Future Perspectives. *J Comp Pure Appl Math*, **4**(1):1-23. Doi: https://doi.org/10.33790/cpam1100124.
Copyright: 2026, This is an open-access article distributed under the terms of the Creative Commons Attribution License 4.0, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

Abstract: As a generalization of classical calculus, fractional calculus provides an effective framework for describing memory and nonlocal effects in complex systems. This survey reviews its historical evolution, theoretical foundations, computational advances, and modern applications from the perspective of applied mathematics. It outlines the establishment of fractional calculus as an independent research field, including its development in China, and discusses how advances in numerical methods have facilitated its applications in physics, engineering, and biology. Four representative cases are examined: anomalous diffusion in a circular comb structure, squeeze-film flow in the knee joint, fractional kinetic equations in statistical mechanics, and time-fractional heat conduction in different geometries. These cases illustrate how fractional-order models characterize anomalous transport, viscoelastic memory, nonlocal particle dynamics, and geometry-dependent heat transfer more effectively than conventional integer-order models in appropriate settings. Recent connections between fractional calculus and machine learning are also reviewed. Finally, the main challenges and potential directions for future interdisciplinary research are discussed.
Keywords: Fractional calculus; Applied mathematics; Modern applications; Fractional-order model

1 INTRODUCTION

Fractional calculus extends classical differentiation and integration to non-integer orders and provides an effective framework for describing memory, hereditary effects, and nonlocal behavior. These properties make fractional-order models particularly useful for complex systems that cannot be adequately represented by conventional integer-order calculus, including viscoelastic materials, anomalous transport, and biological processes. From the perspective of applied mathematics, this paper reviews the historical development, theoretical foundations, and modern applications of fractional calculus. The literature on fractional calculus can be broadly divided into two categories: studies of the historical development of the field and studies of its theoretical foundations. The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order [1] was the first monograph on fractional calculus, published in 1974, and it analyzed the research overview of fractional calculus since its origin. In 1975, Ross [2] published A Brief History and Exposition of the Fundamental Theory of Fractional Calculus in an edited volume. Then he wrote The Development of Fractional Calculus 1695-1900 [3] in 1977. This was the first paper on the development of fractional calculus, which detailed the current status of fractional calculus since it was proposed. In 1993, An Introduction to the Fractional Calculus and Fractional Differential Equations [4] was published by Miller and Ross. This book mainly discusses the development of fractional calculus and fractional differential equations, from the introduction of history to the theoretical discussion. In 2011, Machado et al. [5] summarized important events in the field of fractional calculus since 1974. In 2014, Lazarevic et al. [6] in their paper summarized the brief history of the field and introduced the basic theory of fractional calculus. In 2022, Iqbal M. Batiha et al. [7] reviewed the basic theory of fractional calculus. Although the above studies provide a relatively comprehensive discussion of the historical development and basic theory of fractional calculus, recent advances in numerical methods have not been sufficiently reviewed. Therefore, it is necessary to examine this aspect in greater detail. The development of fractional calculus was initially slow because of its limited practical applications. Mandelbrot's work on fractional dimensions and self-similarity promoted its connection with natural phenomena [8]. Subsequent studies demonstrated its applications in physical systems, tumor modeling, science, and engineering [9-11]. To illustrate its practical value, this paper examines four representative cases: anomalous

diffusion in a circular comb-inward structure, knee-joint squeeze-film flow, fractional kinetic equations in statistical mechanics, and time-fractional heat conduction in different geometries. The remainder of the paper is organized as follows. Section 2 traces the historical development of fractional calculus from its origin to the present. Section 3 examines its development in China and highlights the contributions of Chinese scholars. Section 4 reviews representative modern applications and presents four cases involving anomalous diffusion, knee-joint squeeze-film flow, fractional kinetic equations in statistical mechanics, and time-fractional heat conduction. The integration of fractional calculus with machine learning is discussed in Section 2. Section 5 summarizes and discusses the principal findings, while Section 6 presents the conclusions and future research perspectives.

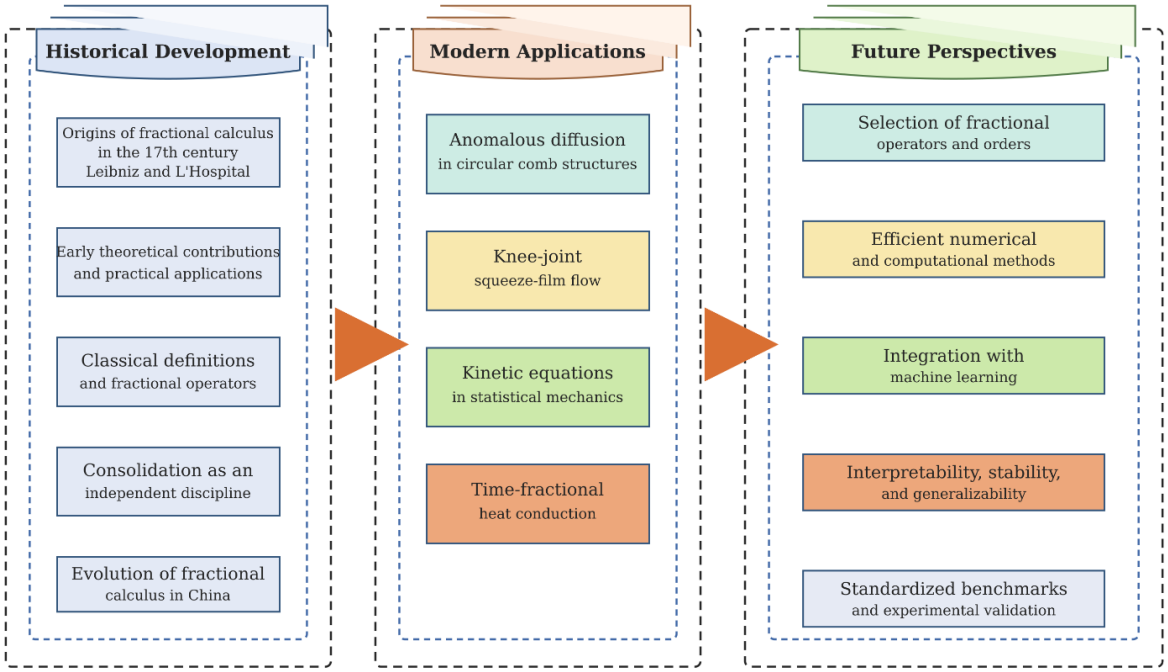


Fig. 1. Overall framework of the review: historical development, modern applications, and future perspectives

2 HISTORY OF DEVELOPMENTS

The study of fractional calculus originated in the late 17th century, with early research primarily focusing on extending the concepts of classical calculus to include non-integer orders and examining the validity of these extensions. For instance, in 1730, Euler mentioned in a letter that for $\frac{d^n p}{dx^n}$ [3]. The question was then raised as to how this ratio

could be defined when n was fractional. Although derivatives of integer order could be obtained by repeated differentiation, this method was not directly applicable when n was fractional. Euler observed that it was possible to simplify the process through the interpolation method. His work laid the groundwork for the development of fractional calculus by subsequent mathematicians. In 1772, Lagrange examined whether the operational properties of integer-order calculus could be extended to arbitrary orders [12]. In the following hundred years, although the concept of fractional order was occasionally mentioned, no substantial results were achieved. Until the early 19th century, fractional order appeared as the number of operations in differential and integral operations. In 1812, Laplace defined differentiation of arbitrary integer orders, provided specific expressions for fractional differentiation operations, and was the first to propose the concept of differentiation of arbitrary order. In 1819, Lacroix provided the first correct answer to the problem raised by Leibniz and L'Hospital, stating that [13]:

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = 2\sqrt{\pi x}. \tag{1}$$

In his monograph on Calculus published in 1819, Lacroix developed the formula for n-th derivative of $y = x^m$, with m being a positive integer:

$$D_x^n y = \frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n}, \quad m \geq n. \tag{2}$$

In 1822, Fourier derived the Fourier definition of any order differential [14]. If

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\alpha)\, d(\alpha) \int_{-\infty}^{+\infty} \cos p(x-\alpha)\, dp, \tag{3}$$

and

$$\frac{d^n}{dx^n} = \cos p(x-\alpha) = p^n \cos\left[p(x-\alpha) + \frac{\pi}{2}\right], \tag{4}$$

for any μ, there is:

$$\begin{aligned} \frac{d^\mu}{dx^\mu} f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} & f(\alpha)\, d(\alpha) \\ & + \int_{-\infty}^{+\infty} p^\mu \cos\left[p(x-\alpha) + \frac{\mu\pi}{2}\right] dp, \end{aligned} \tag{5}$$

and he stated that it could be considered as any real number. During this period, mathematicians mainly obtained conclusions through the study of integer order calculus, ex-

plored the operational properties of fractional order calculus, and obtained some rough conclusions. One of the earliest practical applications of fractional calculus was provided by Abel. In 1823, while solving an integral equation, he considered the following integral transform:

$$K = \int_0^x (x-t)^{-\frac{1}{2}} f(t) dt. \tag{6}$$

By expressing the integral in fractional form and applying the corresponding operator to both sides, Abel obtained the following relation:

$$\frac{d^{\frac{1}{2}} K}{dx^{\frac{1}{2}}} = \sqrt{\pi} f(x). \tag{7}$$

This work demonstrated the inverse relationship between fractional integration and differentiation and showed that they could be regarded as different forms of the same generalized operation, thereby contributing to the early theoretical development of fractional calculus. In 1832, inspired by Fourier and Abel, Liouville proposed the first reasonable definition of fractional derivatives. He pointed out that for integer order derivatives:

$$D^n e^{ax} = a^n e^{ax}, \tag{8}$$

This expression was then extended to derivatives of arbitrary order:

$$D^\alpha e^{ax} = a^\alpha e^{ax}. \tag{9}$$

It was assumed that the function could be expanded as the following series:

$$f(x) = \sum_{n=0}^{\infty} C_n e^{a_n x}, \quad \mathrm{Re}(a_n) > 0, \tag{10}$$

Thus, the following expression for the arbitrary-order derivative was obtained:

$$D^\alpha f(x) = \sum_{n=0}^{\infty} C_n a_n^\alpha e^{a_n x}. \tag{11}$$

This formula is known as Liouville's first formula for a fractional derivative [3]. However, there is still a limitation that the order here cannot be generalized to arbitrary order. To overcome this, Liouville labored to produce a second definition. A definite integral closely related to the Gamma function was first considered:

$$I = \int_0^\infty u^{\beta-1} e^{-xu}\, du, \quad \beta > 0,\ x > 0. \tag{12}$$

From this integral, he derived what is now referred to as Liouville's second formula [6]:

$$D^{\alpha}x^{-\beta} = (-1)^{\alpha}\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}x^{-\alpha-\beta}, \quad \beta > 0. \tag{13}$$

In the second formula, there are still limitations that can only apply to a particular function x^{α}. Although both of the above definitions have not been widely generalized because of some limitations, they play a fundamental role in the establishment of the concept of fractional calculus. In the Mid-19th century, Grunwald and Letnikov independently developed the concept of fractional derivatives. They also developed a numerical method for calculating these derivatives, which came to be known as the Grunwald-Letnikov derivative [6]:

$$\begin{aligned} {}_{a}^{GL}D_{t}^{\alpha}f(t) &= \lim_{h\to 0^{+}} h^{-\alpha}\sum_{r=0}^{\lfloor (t-a)/h \rfloor}(-1)^{r}\binom{\alpha}{r}f(t-rh) \\ &= \sum_{j=0}^{m}\frac{f^{(j)}(a)(t-a)^{j-\alpha}}{\Gamma(j+1-\alpha)} + \frac{1}{\Gamma(m+1-\alpha)}\int_{a}^{t}(t-\tau)^{m-\alpha}f^{(m+1)}(\tau)\,d\tau. \end{aligned} \tag{14}$$

This method approximates the derivative of the function by means of finite difference and can calculate the derivative of the non-integer order. In the latter half of the 19th century, the work of Sonin, Riemann, and Liouville contributed to the establishment of the Riemann-Liouville fractional integral and derivative as fundamental operators of fractional calculus. The Riemann-Liouville fractional integral is defined as [15]:

$${}_{a}^{RL}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_{a}^{t}(t-\tau)^{n-\alpha-1}f(\tau)\,d\tau, \quad (n-1 \le \alpha \le n). \tag{15}$$

Based on this integral operator, the corresponding Riemann-Liouville fractional derivative is defined as:

$$\begin{aligned} {}_{\alpha}^{RL}D_{t}^{\alpha}f(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_{\alpha}^{t}(t-\tau)^{n-\alpha-1}f(\tau)\,d\tau \\ &= \sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)(t-\alpha)^{-\alpha+k}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)}\int_{\alpha}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)\,d\tau \\ &= {}_{\alpha}^{GL}D_{t}^{\alpha}f(t), \end{aligned} \tag{16}$$

where the function $f(t)$ must be $m+1$ times continuously differentiable. Unfortunately, the Riemann-Liouville approach leads to initial conditions containing the limit values of

the Riemann-Liouville fractional derivative at the lower terminal $t = a$, for example:

$$\begin{aligned} \lim_{t\to a} {}_{\alpha}^{RL}D_t^{\alpha-1} f(t) &= b_1, \\ \lim_{t\to a} {}_{\alpha}^{RL}D_t^{\alpha-2} f(t) &= b_2, \\ &\vdots \\ \lim_{t\to a} {}_{\alpha}^{RL}D_t^{\alpha-n} f(t) &= b_n. \end{aligned} \tag{17}$$

Although initial-value problems with such conditions can be solved mathematically, these conditions may be difficult to interpret in practical applications [16]. To overcome the limitations of the Riemann-Liouville fractional derivative, Miroslav Caputo proposed the concept of the Caputo fractional derivative in the 1960s of the 20th century. His work represented an important contribution to fractional calculus. He defined the fractional derivative as follows:

$${}_{a}^{C}D_t^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \quad (n-1 \le \alpha \le n). \tag{18}$$

The Caputo derivative is defined by first taking the regular derivative of the nth order for the function, and then performing the $n - \alpha - 1$ fractional integral. When dealing with problems with initial conditions, this method is more reasonable and natural because it considers the value of the function at the starting point, such as $y'(a)$, $y''(a)$. For Riemann-Liouville and Caputo fractional derivatives, there is such a relationship between them:

$${}_{a}^{RL}D_t^{\alpha} f(t) = {}_{a}^{C}D_t^{\alpha} f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{19}$$

When $f(a) = f\prime(a) = \cdots = f^{(n-1)}(a) = 0$, we have ${}_{a}^{RL}D_t^{\alpha} f(t) = {}_{a}^{C}D_t^{\alpha} f(t)$. In particular, the Caputo derivative shows remarkable utility in solving problems related to physics and engineering. The development of the Riemann-Liouville definition marked a new phase in the study of fractional calculus. At this stage, the differential and integral operators were given unified expressions, and the theory achieved a level of systematization. Subsequently, in order to further expand the theoretical scope of fractional calculus and address other limitations, Marcel Riesz proposed the Riesz fractional derivative based on

the definition of Riemann Liouville fractional derivatives [17]:

$$\begin{aligned}\frac{\partial^a}{\partial|x|^a}u(x) &= -\frac{1}{2\cos(a\pi/2)}\left\{{}_a^{RL}D_t^a u(t) + {}_t^{RL}D_b^a u(t)\right\} \\ &= -\frac{1}{2\cos(a\pi/2)}\left\{\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{u(t)}{(t-\xi)^{\alpha+1-n}}\,d\xi \right. \\ &\qquad \left. +\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b \frac{u(t)}{(\xi-t)^{\alpha+1-n}}\,d\xi\right\}.\end{aligned} \tag{20}$$

Riesz fractional derivative, based on the definition of Fourier transform, provides a new perspective for dealing with fractional calculus problems. Especially in signal processing and wave equation analysis, it has demonstrated its unique advantages. For example, for the space-time Riesz-Caputo fractional advection-diffusion equation, it can be expressed as:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = B_1\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}u(x,t) + B_2\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u(x,t) + f(x,t), \tag{21}$$

where $0 < \alpha \leq 1$, $0 < \beta_1 < 1$, $1 < \beta_2 < 2$ are real parameters. The coefficients B_1 and B_2 are both positive constants [17]. With the widespread application of fractional calculus in physics, mathematicians and physicists have begun to study variable fractional derivatives in order, such as Riemann Liouville variational fractional derivative [16] to address the limitations of traditional fixed order derivatives in describing complex physical phenomena:

$${}_\alpha^{RL}D_t^{\alpha(x)}f(t) = \left[\frac{1}{\Gamma(n-\alpha(x))}\frac{d^n}{d\xi^n}\int_a^\xi (\xi-\tau)^{n-\alpha(x)-1}f(\tau)\,d\tau\right]_{\xi=x}. \tag{22}$$

The order of this derivative can vary over time and space, providing a more flexible mathematical tool. With the deepening of research, the theory of variable fractional order derivatives has been developed and improved, providing more accurate solutions for dealing with practical physics problems and expanding the application scope of fractional calculus. It is evident that, with the deepening of traditional fractional calculus theory in practical applications, the development of variable fractional calculus theory has emerged from the needs of these practical applications. Clearly, in the process of cross integration with other disciplines, the development of fractional calculus has also been further promoted. In the 1960s, in order to facilitate the handling of initial conditions, Caputo

extended the time fractional derivative from the fractional derivative:

$$ {}_a^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} u_\tau^{(n-1)}(x,\tau)\, d\tau, \quad (n-1 \le \alpha \le n). \tag{23} $$

Furthermore, based on this, the distribution order time fractional derivative was extended:

$$ D_t^w u(x,t) = \int_0^\beta w(\alpha)\, {}_a^C D_t^\alpha u(x,t)\, d\alpha, \quad 0 < \beta \le 1, \tag{24} $$

where $\omega(\alpha) \ge 0$, $\int_0^\beta w(\alpha)\, d\alpha = c_0 > 0$.This makes Caputo fractional derivative particularly useful in physics and engineering problems. The case in point is, for the two-dimensional time distribution order diffusion equation, it can be expressed as [18]:

$$ \begin{cases} D_t^\nu u(x,t) - \nu \Delta u(x,t) + c(x)u(x,t) = f(x,t), & \text{for } x = (x_1, x_2) \in \Omega,\ 0 < t \le T, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega, \\ u(x,t)|_{\partial\Omega} = 0, & \text{for } 0 \le t \le T. \end{cases} \tag{25} $$

In 1974, the first International Conference on Fractional Calculus was held in the United States, and the first part of the monograph on fractional calculus was published in the same year. These events marked the beginning of the systematic study of fractional calculus theory and heralded a research boom in the field. Additionally, during this period, the transformation relationships between the Grunwald-Letnikov, Riemann-Liouville, and Caputo definitions of fractional derivatives were established. At the beginning of the 21st century, with the advancement of computing technology, the study of fractional calculus entered a new stage. Researchers began to explore deeper into the non-local properties of fractional calculus operators, finding that these properties tend to make fractional-order models more reflective of real-world physical processes than traditional integer-order models. The non-locality of fractional operators is the key difference from traditional local operators [19]. This property makes fractional models particularly suitable for describing complex systems with long memory properties, such as viscoelastic materials. Considering a one-dimensional body and assuming total stored energy depending not only upon the local strain but also upon a fractional derivative of the strain, an elastic model with non-local stress-strain behavior was introduced. Fractional calculus provides a natural framework for describing non-local constitutive relations and requires no assumptions for the interval of non-local influence. However, owing to their nonlocal nature, fractional derivatives depend on the values of a function over an interval rather than only at a single point, resulting in greater computational and memory costs than integer-order derivatives.

Therefore, specialized numerical approximations are required. For the Caputo fractional derivative, the derivative in the integrand can be approximated using finite differences. The L1 approximation for the right-sided Caputo fractional derivative is given by [20]:

$$ {}_{x}^{C}D_{b}^{\alpha} f(t_n) \approx \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{N-n-1} \left[f(t_{n+j}) - f(t_{n+j+1}) \right] \left[(j+1)^{1-\alpha} - j^{1-\alpha} \right], \tag{26} $$

The L2 scheme is:

$$ \begin{aligned} {}_{x}^{C}D_{b}^{\alpha} f(t_n) \approx \frac{h^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{N-n-1} & \left[f(t_{n+j-1}) - 2f(t_{n+j}) + f(t_{n+j+1}) \right] \\ & \left[(j+1)^{2-\alpha} - j^{2-\alpha} \right]. \end{aligned} \tag{27} $$

With the continuous advancement of computing technology, there has been a significant shift in how we approach and solve problems in fractional calculus. Software and tools specially developed to meet complex calculation needs have become the key to solving fractional calculus problems. In the field of fractional calculus control research, since 1990, four main toolboxes have been developed: CRONE, Ninteger, FOTF, and FOMCOM. These tools dramatically improve the accuracy and efficiency of solving, allowing researchers to solve previously unsolvable problems. By this period, fractional calculus had become a multidisciplinary research field, with its theory and applications constantly developing and improving. With the development of new theories and the expansion of application fields, fractional calculus will continue to play an important role in modern scientific research. Numerical methods for fractional calculus have developed rapidly in recent years because fractional differential equations often lack closed-form analytical solutions [20]. Finite difference methods provide convenient approximations of fractional derivatives on structured grids, whereas finite element methods are well suited to problems involving complex geometries. More recently, machine-learning methods have provided new approaches for solving forward and inverse fractional differential equations [21]. These developments have expanded the practical application of fractional-order models in science and engineering. In recent years, the intersection of fractional calculus with machine learning and neural networks has emerged as a vibrant research frontier. Fractional derivatives, with their inherent ability to capture memory effects and nonlocal dependencies, have been integrated into various machine learning frameworks to enhance their modeling capabilities. Raubitzek et al. [22] provided a comprehensive review of com-

bined approaches between fractional derivatives and machine learning, categorizing them into three main directions: preprocessing, machine learning with fractional dynamics, and optimization. They demonstrated that fractional derivatives can serve as powerful preprocessing and feature augmentation techniques, improve physically informed machine learning, and enhance hyperparameter optimization. Within the domain of machine learning, fractional calculus has been employed in multiple ways. Joshi et al. [23] conducted a comparative study of integer-order and fractional-order artificial neural networks for mathematical function generation, showing that fractional-order networks exhibit superior performance in capturing long-range dependencies. Shekarpaz et al. [24] introduced a splitting PINN approach for solving both integer and fractional-order neuron models, employing an L1 scheme for discretizing fractional derivatives to improve accuracy and efficiency. Nuugulu et al. [25] developed a PINN framework incorporating the Grünwald-Letnikov fractional derivative through a stable matrix formulation to solve the time-space fractional Black-Scholes equation, eliminating mesh discretization constraints typical of finite-difference methods. Coelho et al. [26] presented a comprehensive survey of fractional calculus neural network-based computer vision techniques, covering applications such as denoising, enhancement, object detection, segmentation, and restoration, highlighting how fractional derivatives capture complex spatial and temporal relationships in images and videos. Recent studies have further improved the accuracy and computational efficiency of machine-learning methods for fractional differential equations. Vellappandi and Lee [27] developed a physics-informed neural fractional differential-equation framework that combines fractional calculus with neural differential-equation modeling. Zhang et al. [28] proposed spectral-fPINNs, in which spectral discretization based on Jacobi polynomials is incorporated into fractional physics-informed neural networks to improve computational efficiency. Ma et al. [29] developed an adaptive fractional physics-informed neural network for solving forward and inverse problems of anomalous heat conduction in functionally graded materials. These studies demonstrate the continuing development of machine-learning methods for fractional-order modeling and extend their applications to complex forward and inverse problems. While the studies discussed above reflect recent international advances in the integration of fractional calculus with machine learning, the development of fractional calculus has also been shaped by sustained research efforts in different countries. The following section focuses on the development of fractional calculus in China and its contribution to the broader advancement of the field.

3 EVOLUTION IN CHINA

Although fractional calculus was proposed after classical calculus, it has a history of more than 300 years [30]. Since fractional calculus research developed relatively late in China, the amount of related research was initially limited. From the late twentieth century onward, fractional calculus gradually attracted attention from Chinese researchers in both theoretical and applied mathematics [20-21]. In the early 2000s, an increasing number of papers and monographs began to appear, reflecting the gradual development of the field. A bibliometric analysis of 2,854 Chinese publications showed that fractional-order research experienced relatively slow development from 2001 to 2010, rapid growth from 2010 to 2014, and steady development from 2014 to 2019, followed by continued growth in 2020 [31]. During this process, fractional calculus was increasingly applied to control theory, signal processing, anomalous diffusion, biomedicine, and materials science [20,30]. In recent years, related research has further expanded into information technology, machine learning, and intelligent optimization, demonstrating considerable potential for future interdisciplinary development [32]. Selected representative contributions are summarized in Table 1.

Table 1. Selected representative contributions to fractional calculus research by Chinese scholars

Year	Representative study	Research area	Main contribution
2004	Chen and Holm [33]	Fractional wave modeling	Developed fractional Laplacian time-space models to describe frequency-dependent attenuation in lossy media.
2009	Deng [34]	Numerical analysis	Established a finite element method for the space- and time-fractional Fokker-Planck equation and analyzed its stability and convergence.
2015	Li and Zeng [20]	Computational fractional calculus	Systematically prese derivatives, and differential equations.
2018	Sun et al. [30]	Interdisciplinary applications	Summarized representative applications of fractional calculus in transport, mechanics, control, signal processing, biology, and materials science.

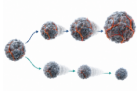
Year	Representative study	Research area	Main contribution
2025	Zhang et al. [28]	Fractional calculus and machine learning	Developed spectral fractional physics-informed neural networks for solving fractional partial differential equations.

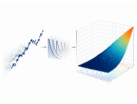
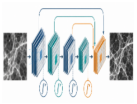
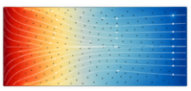
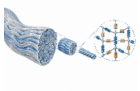
As summarized in Table 1, fractional calculus research in China has progressed from theoretical modeling and conventional numerical methods to interdisciplinary applications and machine-learning-based solvers. These developments have improved the analysis and computation of fractional differential equations and strengthened the connection between mathematical theory and practical applications. The historical and regional developments described above have provided the theoretical and computational foundations for the growing application of fractional calculus to practical scientific and engineering problems.

4 MODERN APPLICATIONS

In recent years, fractional calculus has been widely applied in various fields such as physics, biology, and engineering, as its non-integer order derivatives and integrals enable more accurate descriptions of complex real-world phenomena involving memory and nonlocality [9]. Table 2 summarizes several representative applications of fractional calculus and the corresponding fractional-order models or computational methods. To illustrate these applications in greater detail, this section presents four representative case studies: anomalous diffusion on a circular comb-inward structure, squeeze-film flow in a knee joint, fractional kinetic equations in statistical mechanics, and time-fractional heat conduction in different geometries.

Table 2. Representative modern applications of fractional calculus.

Field	Representative application	Fractional model or method	Representative study	Schematic illustration
Biology and medicine	Tumor-growth and treatment-response modeling	Fractional-order biological models	Vieira et al. [10]	

Field	Representative application	Fractional model or method	Representative study	Schematic illustration
Finance	Pricing under time-space fractional dynamics	Fractional Black-Scholes equation and physics-informed neural network	Nuugulu et al. [25]	
Computer vision	Image denoising, enhancement, detection, and segmentation	Fractional operators and fractional-calculus-based neural networks	Coelho et al. [26]	
Heat transfer and materials	Anomalous heat conduction in functionally graded materials	Adaptive fractional physics-informed neural network	Ma et al. [29]	
Control engineering	Control of nonlinear systems such as helicopters and inverted pendulums	Fractional-order PID controller and fractional-order actor-critic algorithm	Shalaby et al. [35]	
Biomechanics	Viscoelastic response of tendons and soft tissues	Tempered fractional-order viscoelastic model	Wang and Guo [36]	

4.1 Case Study 1: Anomalous Diffusion on a Circular Comb-Inward Structure

Anomalous diffusion on comb-like structures is commonly observed in biological systems such as calcium transport in spiny dendrites and tumor cell migration. To model this phenomenon, Liu et al. [37] investigated diffusion dynamics on a circular comb-inward structure with different radial and tangential mobilities. This structure consists of a circular backbone and radial branches, where diffusion occurs in the tangential direction only at the circular backbone. The Scott-Blair fractional memory model was employed to incorporate the trapping process:

$$\begin{aligned} J_r &= -D_r \, {}^{RL}D_t^{1-\alpha} \left[\frac{\partial P}{\partial r} \right], \\ J_\theta &= -D_\theta \, {}^{RL}D_t^{1-\alpha} \left[\int \frac{1}{r} \frac{\partial P}{\partial \theta} \delta(r - r_0) \, dr \right], \end{aligned} \tag{28}$$

where ${}^{RL}D_t^{1-\alpha}$ denotes the Riemann–Liouville time-fractional derivative, defined as:

$${}^{RL}D_t^{1-\alpha} P(r,\theta,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{P(r,\theta,\tau)}{(t-\tau)^{1-\alpha}} \, d\tau, \quad 0 < \alpha < 1. \tag{29}$$

The results presented in Fig. 2 indicate that the fractional order α has a significant influence on particle diffusion behavior. As α decreases, the peak of the marginal distribution rises while both sides decline, and the mean-square displacement exhibits sublinear growth with a gradually decreasing growth rate, indicating stronger memory effects. This case study effectively demonstrates the capability of fractional calculus to capture memory-dependent anomalous diffusion in complex geometric structures.

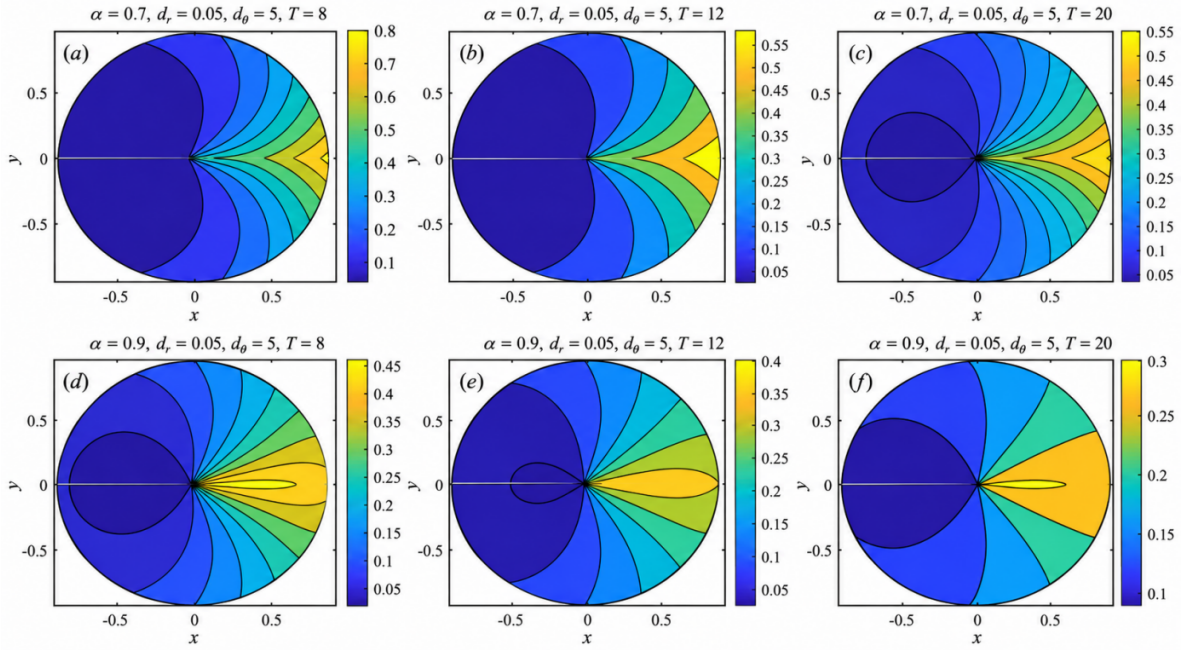


Fig. 2. Time evolution of the contour maps of P for different values of fractional order α. Adapted from Liu et al. [37].

4.2 Case Study 2: Squeeze-Film Flow in a Knee Joint

Effective joint lubrication is essential for the health of articular cartilage. To investigate this biomechanical process, Yang et al. [38] employed physics-informed neural networks to solve the squeeze-film flow problem in a knee joint. In this model, the femoral component is treated as a rigid sphere, the cartilage as a porous medium, and the synovial fluid as a fractional Maxwell fluid. To describe the viscoelastic behavior of the synovial fluid, the fractional Maxwell constitutive model is adopted, in which the Caputo fractional derivative is used to characterize the stress-strain relationship. The governing

equation for the squeeze-film flow can be expressed as:

$$\begin{aligned} \frac{dh_{\min}}{dt} &= \frac{1}{12r}\frac{\partial}{\partial r}\left[rF(h)\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\frac{\partial p}{\partial r}\right], \\ F(h) &= \frac{h^2(\alpha h^2+4\sqrt{k}\,h+6\alpha k)+12k(\alpha h+\sqrt{k})}{\alpha h+\sqrt{k}}. \end{aligned} \tag{30}$$

where the Caputo fractional derivative used to describe the viscoelastic behavior of the synovial fluid is defined as:

$${}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} f\prime(\tau)\,d\tau, \quad 0<\alpha<1, \tag{31}$$

Fig. 3 presents the radial velocity distributions for different fractional orders α and permeability values k. The results indicate that cartilage permeability is a key factor determining lubrication performance. Higher permeability accelerates synovial fluid loss, leading to a rapid decline in film thickness and fluid load-bearing capacity, thereby exacerbating cartilage contact and wear. Under the same permeability, a larger α results in a flatter velocity profile near the femoral surface, indicating a stronger shear-thinning effect. For a fixed α, increasing k significantly reduces the velocity magnitude in both the fluid and porous layers, reflecting greater fluid infiltration into the cartilage.

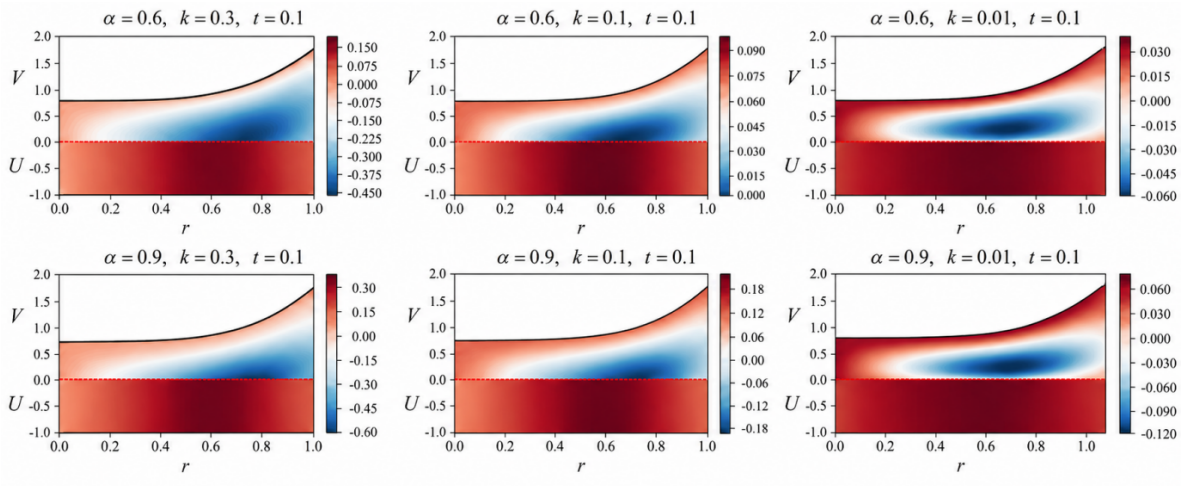


Fig. 3. Velocity field distributions of the squeeze-film flow in the knee joint. Adapted from Yang et al. [38].

4.3 Case Study 3: Fractional Calculus in Statistical Mechanics: A Theoretical Foundation

Fractional calculus provides a mathematical framework for extending the fundamental kinetic equations of statistical mechanics to systems involving memory and spatial nonlocality. Magin and Lenzi [39] reviewed fractional modifications of the Boltzmann, Fokker-Planck, and Langevin equations. In these formulations, time-fractional operators are generally associated with long-memory and waiting-time effects, whereas space-fractional operators describe nonlocal transport and heavy-tailed particle displacements. The fractional Boltzmann equation, originally formulated by Nonnenmacher and Nonnenmacher, replaces the integer-order time derivative with a Riemann-Liouville fractional integral, introducing a fading memory that decays as an inverse power law. For a single-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$, the equation takes the form:

$$\partial_t f(\mathbf{x}, \mathbf{v}, t) = \frac{t^{\nu-1}}{\Gamma(\nu)} F(\mathbf{x}, \mathbf{v}, 0) + \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} F\prime(\mathbf{x}, \mathbf{v}, \tau)\, d\tau, \tag{32}$$

where $F\prime$ includes the nonlinear collision integral, the streaming term, and external forces. This formulation ensures that the standard Boltzmann equation is recovered in the limit $\nu \to 1$. As shown in Fig. 4, the Boltzmann, Fokker-Planck, and Langevin equations connect complementary descriptions of particle dynamics with Brownian motion, Lévy flights, fractional Brownian motion, and Ornstein-Uhlenbeck processes. Their fractional extensions provide mechanisms for representing anomalous transport: temporal fractional operators describe long-memory and waiting-time effects, whereas spatial fractional operators characterize nonlocal motion and heavy-tailed particle displacements. This framework broadens the application of kinetic theory to heterogeneous and multiscale systems in which classical Markovian models may be inadequate.

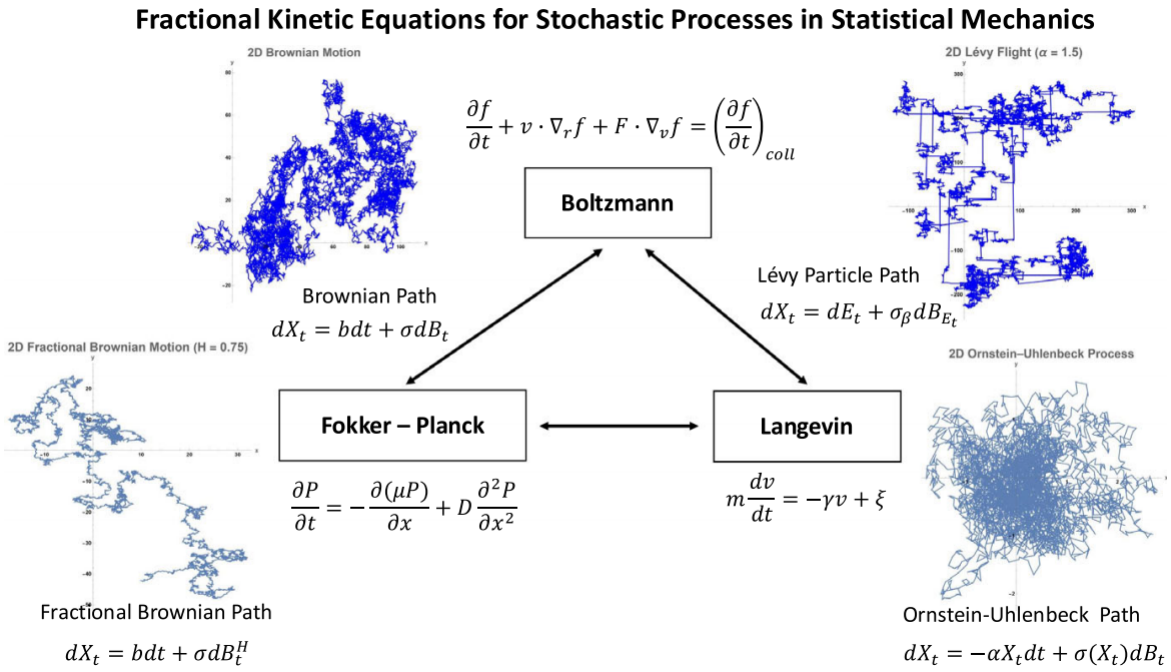


Fig. 4. Relationships among the Boltzmann, Fokker-Planck, and Langevin equations and representative stochastic processes. Adapted from Magin and Lenzi [39].

4.4 Case Study 4: Time-Fractional Heat Conduction in Different Geometries

Fractional calculus has been applied to heat-transfer problems to describe thermal processes involving memory and transitions between diffusive and wave-like behavior. Terpák [40] developed a general one-dimensional time-fractional diffusion-wave model applicable to a plane wall, a long cylinder, and a sphere. The governing equation can be expressed in a unified form as

$$\rho c_p \frac{\partial^\alpha T(r,t)}{\partial t^\alpha} = \frac{1}{r^\gamma}\frac{\partial}{\partial r}\left(r^\gamma \lambda \frac{\partial T(r,t)}{\partial r}\right), \quad 0 < \alpha \leq 2, \tag{33}$$

where $T(r,t)$ denotes temperature, ρ is the material density, c_p is the specific heat capacity, λ is the thermal conductivity, and α is the order of the time-fractional derivative. The geometry parameter γ takes the values 0, 1, and 2 for a plane wall, a long cylinder, and a sphere, respectively. Thus, the same fractional model can describe heat conduction in different one-dimensional geometries by adjusting only the geometry parameter. The model was numerically solved using finite-difference schemes based on the Grünwald-Letnikov fractional derivative. Fig. 5 compares the temperature evolution in a plane

wall for $\alpha = 0.5$, 1 and 1.5. When $\alpha = 0.5$, the temperature initially increases rapidly and subsequently approaches equilibrium more slowly. The case $\alpha = 1$ corresponds to the classical heat-diffusion equation, whereas $\alpha = 1.5$ exhibits characteristics of both diffusion and wave propagation. These results demonstrate that the fractional order controls the temporal characteristics of heat transfer and enables a unified description of diffusion-dominated and diffusion-wave processes in materials with different geometries.

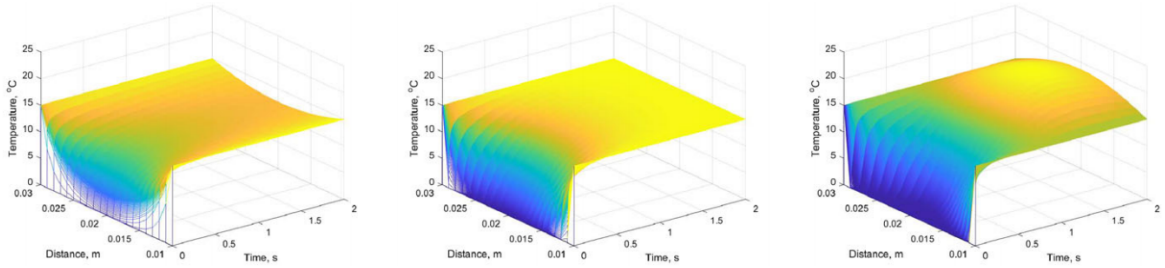


Fig. 5. Temperature evolution in a plane wall for different fractional orders $\alpha = 0.5$,1, and 1.5. Adapted from Terpák [40].

5 RESULTS AND DISCUSSION

This review shows that fractional calculus has evolved from an early generalization of classical differentiation and integration into an important mathematical framework for describing memory and nonlocal behavior. Its development in China followed the broader international trend and gradually expanded from theoretical studies to engineering and interdisciplinary applications. Advances in numerical analysis and computational technology have further promoted the practical implementation of fractional-order models. The four representative cases demonstrate the applicability of fractional calculus to different problems. The Scott-Blair model describes anomalous diffusion in circular comb structures, while the fractional Maxwell model characterizes the memory-dependent rheological behavior of knee-joint squeeze-film flow. Fractional kinetic equations extend statistical mechanics to non-Markovian and nonlocal particle dynamics, whereas the time-fractional diffusion-wave equation provides a unified description of heat conduction in different geometries. These cases demonstrate the flexibility of fractional-order modeling, although the operators and orders must be selected according to the physical mechanisms of specific problems.

6 CONCLUSIONS AND FUTURE PERSPECTIVES

Fractional calculus provides effective mathematical tools for modeling complex systems involving memory, anomalous transport, viscoelasticity, and spatial nonlocality. The historical development and representative applications reviewed in this paper indicate that its growing importance in applied mathematics is closely associated with advances in numerical methods and computational technology. Its integration with machine learning has also created new possibilities for solving and identifying fractional-order systems. Future research should establish more systematic criteria for selecting fractional operators and determining their orders from physical mechanisms and experimental data. More efficient numerical algorithms are needed to reduce the computational and memory costs of multidimensional and long-time simulations. In addition, standardized benchmark problems and further experimental validation will be essential for improving the accuracy, interpretability, stability, and generalizability of fractional-order models and promoting their reliable application in science and engineering.

Funding

This research was supported by the National Natural Science Foundation of China 12102032, Beijing University of Civil Engineering and Architecture 2025 University Education Science Research Project, Y2504; Research Project of the Mathematics Research Branch of Beijing Association of Higher Education/Beijing Interdisciplinary Science Society SXJC-2024-019; Beijing University of Civil Engineering and Architecture, 2026 "AI+Teaching" Pilot Course Construction Project A2026018.

Conflict of interest

The authors declare no conflict of interest.

References

[1]. Oldham, K., & Spanier, J. (1974). The fractional calculus theory and applications of differentiation and integration to arbitrary order. Elsevier.

[2]. Ross, B. (1975). A brief history and exposition of the fundamental theory of fractional

calculus. In B. Ross (Ed.), *Fractional calculus and its applications*. Springer-Verlag.

[3]. Ross, B. (1977). The development of fractional calculus 1695 – 1900. *Historia Mathematica*, **4**(1), 75-89.

[4]. Miller, K. S., & Ross, B. (1993). An introduction to the fractional calculus and fractional differential equations. Wiley.

[5]. Machado, J. T., Kiryakova, V., & Mainardi, F. (2011). Recent history of fractional calculus. *Communications in Nonlinear Science and Numerical Simulation*, **16**(3), 1140-1153.

[6]. Lazarević, M. P., Rapaić, M. R., Šekara, T. B., Mladenov, V., & Mastorakis, N. (2014). Introduction to fractional calculus with brief historical background. Advanced topics on applications of fractional calculus on control problems, *system stability and modeling*, 3-16.

[7]. Batiha, I. M., Alshorm, S., Jebril, I. L. H., & et al. (2022). A brief review about fractional calculus. *International Journal of Computer Mathematics*, **15**(4), 40-56.

[8]. Mandelbrot, B. B., & Aizenman, M. (1979). Fractals: form, chance, and dimension.

[9]. Boulaaras, S., Jan, R., & Pham, V. T. (2023). Recent advancement of fractional calculus and its applications in physical systems. *The European Physical Journal Special Topics*, **232**(14), 2347-2350.

[10]. Vieira, L. C., Costa, R. S., & Valério, D. (2023). An overview of mathematical modelling in cancer research: Fractional calculus as modelling tool. *Fractal and Fractional*, **7**(8), 2-22.

[11]. Jacob, J. S., Priya, J. H., & Karthika, A. (2020). Applications of fractional calculus in science and engineering. *Critical Reviews in Biotechnology*, **7**(13), 4385-4394.

[12]. Pertz, G. H., & Gerhardt, C. J. (1849). Leibnizens gesammelte werke, lebinizens mathematische schriften, erste Abtheilung, Band II. Briefwechsel zwischen Leibniz, 301-302.

[13]. Lacroix, S. F. (1797). Traité du calcul différentiel et du calcul intégral. JBM Duprat.

[14]. Fourier, J. B. J. (1878). The analytical theory of heat. The University Press.

[15]. Lovoie, J. L., Osler, T. J., & Tremblay, R. (1976). Fractional derivatives and special functions. *Society for Industrial and Applied Mathematics*, **18**(2), 240-268.

[16]. Podlubny, I. (1998). Fractional differential equations: An introduction to fractional

derivatives, fractional differential equations, to methods of their solution and some of their applications. Elsevier.

[17]. Shen, S., Liu, F., & Anh, V. (2011). Numerical approximations and solution techniques for the space-time Riesz-Caputo fractional advection-diffusion equation. *Numerical Algorithms*, 56, 383-403.

[18]. Ren, J., & Chen, H. (2019). A numerical method for distributed order time fractional diffusion equation with weakly singular solutions. *Applied Mathematics Letters*, 96, 159-165.

[19]. Lazopoulos, K. A. (2006). Non-local continuum mechanics and fractional calculus. *Mechanics Research Communications*, **33**(6), 753-757.

[20]. Li, C., & Zeng, F. (2015). Numerical methods for fractional calculus. CRC Press.

[21]. Machado, J. A. T., Kiryakova, V., Kochubei, A., & et al. (2019). Recent history of the fractional calculus: Data and statistics. In Handbook of Fractional Calculus with Applications.

[22]. Raubitzek, S., Mallinger, K., & Neubauer, T. (2023). Combining fractional derivatives and machine learning: A review. *Entropy*, **25**(1), 1-22.

[23]. Joshi, M. P., Bhosale, S., & Vyawahare, V. A. (2024). Comparative study of integer-order and fractional-order artificial neural networks: Application for mathematical function generation. e-Prime-Advances in Electrical Engineering, Electronics and Energy, 8, 100601.

[24]. Shekarpaz, S., Zeng, F., & Karniadakis, G. E. (2024). Splitting physics-informed neural networks for inferring the dynamics of integer- and fractional-order neuron models. *Communications in Computational Physics*, **35**(1), 1-37.

[25]. Nuugulu, S. M., Patidar, K. C., & Tarla, D. T. (2025). An efficient physics-informed neural network solution to the time-space fractional Black-Scholes equation. *Operations Research Forum*, **6**(4), 1-32.

[26]. Coelho, C., Costa, M. F. P., & Ferrás, L. L. (2024). Fractional calculus meets neural networks for computer vision: A survey. AI, 5(3), 1391-1426.

[27]. Vellappandi, M., & Lee, S. (2025). Physics-informed neural fractional differential equations. *Applied Mathematical Modelling*, 145, 116127.

[28]. Zhang, T., Zhang, D., Shi, S., & Guo, Z. (2025). Spectral-fPINNs: Spectral method based fractional physics-informed neural networks for solving fractional partial dif-

ferential equations. *Nonlinear Dynamics*, 113, 12565-12588.

[29]. Ma, X., Qiu, L., Zhang, B., Wu, G., & Wang, F. (2025). Adaptive fractional physics-informed neural networks for solving forward and inverse problems of anomalous heat conduction in functionally graded materials. *International Journal of Heat and Mass Transfer*, 236, 126393.

[30]. Sun, H., Zhang, Y., Baleanu, D., Chen, W., & Chen, Y. (2018). A new collection of real world applications of fractional calculus in science and engineering. *Communications in Nonlinear Science and Numerical Simulation*, 64, 213-231.

[31]. Yang, Y., Lv, K., Xue, J., & Huang, X. (2021). A bibliometric analysis and visualization of fractional order research in China over two decades (2001-2020). *Journal of Mathematics*, 7996776.

[32]. Fernandez, E., Huilcapi, V., Birs, I. R., & Cajo, R. (2025). The role of fractional calculus in modern optimization: A survey of algorithms, applications, and open challenges. *Mathematics*, **13**(19), 1-34.

[33]. Chen, W., & Holm, S. (2004). Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency power-law dependency. *The Journal of the Acoustical Society of America*, **115**(4), 1424-1430.

[34]. Deng, W. (2009). Finite element method for the space and time fractional Fokker-Planck equation. *SIAM Journal on Numerical Analysis*, **47**(1), 204-226.

[35]. Shalaby, R., El-Hossainy, M., Abo-Zalam, B., & Mahmoud, T. A. (2023). Optimal fractional-order PID controller based on fractional-order actor-critic algorithm. *Neural Computing and Applications*, 35, 2347-2380.

[36]. Wang, X., & Guo, J. (2025). Fractional-order viscoelastic model for tendons with multilevel self-similar structures. *Applied Mathematical Modelling*, 147, 116222.

[37]. Liu, C., Fan, Y., & Lin, P. (2020). Numerical investigation of a fractional diffusion model on circular comb-inward structure. *Applied Mathematics Letters*, 100, 106053.

[38]. Yang, Y., Liu, C., & Wei, Z. (2026). Solving fractional squeeze-film flow in the knee joint with physics-informed neural networks. *Computer Sciences & Mathematics Forum*, **14**(1), 1-9.

[39]. Magin, R. L., & Lenzi, E. K. (2026). Fractional calculus modifications to the kinetic equations of statistical mechanics. *Fractional Calculus and Applied Analysis*, 29, 1227-1286.

[40]. Terpák, J. (2023). General one-dimensional model of the time-fractional diffusion-wave equatioYang, Y., Liu, C., & Wei, Z. (2026). Solving fractional squeeze-film flow in the knee joint with physics-informed neural networks. *Computer Sciences & Mathematics Forum*, **14**(1), 6.

[41]. Magin, R. L., & Lenzi, E. K. (2026). Fractional calculus modifications to the kinetic equations of statistical mechanics. *Fractional Calculus and Applied Analysis*, 29, 1227-1286.

[42]. Terpák, J. (2023). General one-dimensional model of the time-fractional diffusion-wave equation in various geometries. *Fractional Calculus and Applied Analysis*, 26, 599-618.