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Fusion of boundary surface-link and ribbonness

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ABSTRACT

A boundary surface-link of at least two components in the 4-sphere is proved to be a ribbon surface-link if the surface-link obtained from it by surgery along a local S-nontrivial fusion 1-handle system is a ribbon surface-link. This paper presents a detailed proof of this result, including an appendix, although a similar result was previously published with a portion of the proof missing. The result is stated using the concept of a "local fusion 1-handle", which is more precise than the earlier concept of a "pairwise fusion 1-handle".

Keywords: Boundary surface-link, Ribbon surface-link, Local S-nontrivial fusion, Disk basis.
Mathematics Subject Classification 2020: 57K45; 57K40

1. Introduction

This paper presents a detailed version (including an appendix) on the previous result [1, Theorem 1.1], which was published with a portion of the manuscript missing. The result is stated using the concept of a "local fusion 1-handle", which is more precise than the earlier concept of a "pairwise fusion 1-handle". A *surface-link of r components* is a closed oriented surface F of r connected components $F_i\,(i=1,2,\ldots,r)$ smoothly embedded in the 4-sphere S^4. It is called a *surface-knot* when $r=1$, and a *sphere-link* or an S^2*-link* when F consists of 2-spheres. A surface-link F of $r(\geq 2)$ components $F_i\,(i=1,2,\ldots,r)$ is a *boundary surface-link* if there is a system V of disjoint compact connected oriented 3-manifolds $V_i\,(i=1,2,\ldots,r)$ smoothly embedded in S^4 with $\partial V_i = F_i$, where V_i is called a *Seifert hypersurface* of F_i. The system V is called a *disjoint Seifert hypersurface system* of the boundary surface-link F. A *1-handle system* on a surface-link F is a system h of disjoint 1-handles $h_j\,(j=1,2,\ldots,s)$ on F smoothly embedded in S^4. Let $F(h)$ be the surface-link obtained from F by surgery along a 1-handle system h. The equivalence on a surface-link is considered setwise, so that the equivalence on the surface-link $F(h)$ is independent of a framing of the 1-handle h and determined only by the core arc α of h up to homotopies of the interior of α in $S^4 \setminus F$, [2]. For a subsystem h_1 of a 1-handle system h, let $h_1^c = h \setminus h_1$ which induces a 1-handle system on the surface-link $F(h_1)$ as well as a 1-handle system on F. A *ribbon surface-link* is the surface-link $O(h)$ obtained from a trivial S^2-link O by surgery along a 1-handle system h, [3, 4]. A system h of 1-handles $h_j\,(j=1,2,\ldots,s)$ on F is a *fusion 1-handle system* if $F(h)$ has just $r-s(\geq 1)$ connected components, where each 1-handle h_j is called a *fusion 1-handle*, and a *self 1-handle system* if each 1-handle h_j is a self 1-handle, that is, a 1-handle on a connected component of F. A special feature of a boundary surface-link F is the existence of a self 1-handle system β on F embedded in a disjoint Seifert hypersurface system V for F such that the closed complement $V(\beta) = \mathrm{cl}(V \setminus \beta)$ is a handlebody system and thus, the surface-link $F(\beta)$ is a trivial surface-link in S^4, [2]. For a fusion 1-handle h on a boundary surface-link F with a disjoint Seifert hypersurface system V in S^4, let F_h be the surface-link with two components of F to which the 1-handle h attaches, and V_h the disjoint Seifert hypersurface system for F_h belonging to V. Let $F_h^c = F \setminus F_h$ and $V_h^c = V \setminus V_h$. The following definition is made for a fusion 1-handle on a boundary surface-link.

Definition. Let F be a boundary surface-link in S^4. A fusion 1-handle system h on F is *local* if there is a disjoint Seifert hypersurface system U of F such that every 1-handle belonging to h meets just two components of U. A local fusion 1-handle h_1 on F is *surgically nontrivial*, or simply *S-nontrivial* if $F(\gamma)(h_1)$ is a nontrivial surface-link for a self 1-handle system γ on F embedded into U and disjoint from h such that $U(\gamma)$ is a handlebody system for a disjoint Seifert hypersurface system U for F used for the locality of h_1. Otherwise, h_1 is *S-trivial*. A local fusion 1-handle system h on F is *S-nontrivial* if every 1-handle on F belonging to h is S-nontrivial for a disjoint Seifert hypersurface system U of F used for the locality of h and a self 1-handle system γ on F embedded into U and disjoint from h such that $U(\gamma)$ is a handlebody system.

The following theorem is the main result.

Theorem 1.1. Let F be a boundary surface-link of $r(\geq 2)$ components in S^4. If the surface-link $F(h)$ for a fusion 1-handle system h on F consisting of local S-nontrivial fusion 1-handles is a ribbon surface-link, then the surface-link F is a ribbon surface-link.

The proof of Theorem 1.1 is given in Section 4. There are many non-ribbon boundary surface-links F with two or more components in S^4 such that the surface-link $F(h)$ for a local *S-trivial* fusion 1-handle h on F is a ribbon surface-link. For example, the S^2-link L consisting of the boundary S^2-link $K \times \partial I$ of a normal trivial I-bundle $K \times I$ of a non-ribbon S^2-knot K in S^4 and a splitted trivial S^2-link O is a non-ribbon boundary S^2-link such that the S^2-link $L(h)$ for a local S-trivial fusion 1-handle h on L obtained by thickening a I-fiber is a trivial S^2-link, because the 3-manifold $\mathrm{cl}(K \times I \setminus h)$ is a 3-ball, where $I = [0, 1]$. A surface-link F obtained from this non-ribbon S^2-link L by surgery along every trivial self 1-handle system disjoint from h is a non-ribbon boundary surface-link in S^4 such that h is a local S-trivial fusion 1-handle on F with $F(h)$ a trivial surface-link, where the non-ribbonness of F is shown from [5].

Unless otherwise stated, isotopies of S^4 mean smooth isotopies of S^4 with 4-ball supports, and orientation-preserving diffeomorphisms of S^4 mean orientation-preserving diffeomorphisms of S^4 with 4-ball supports.

2. Disk basis for trivial surface-link and handlebody system in 4-sphere

For a surface-link F in S^4, a *loop-pair* on F is a pair of oriented simple loops on F of geometric intersection number 1, a *loop-pair system* on F is a system of disjoint loop-pairs on F, and a *loop basis* of F is a loop-pair system on F representing a basis of $H_1(F; Z)$. A simple loop c on a surface-link F is *spin* if c bounds a possibly non-orientable immersed surface C in S^4 with $C \cap F_1 = c$ for the component F_1 of F containing c such that the self Z_2-intersection number of C in S^4 with respect to the surface-framing on F_1 is 0, [6]. This is equivalent to saying that the Z_2-quadratic function $q : H_1(F_1; Z_2) \to Z_2$ associated with the surface-knot F_1 in S^4 sends the Z_2-homology class of c to the zero. Note that the present definition of a spin loop on a surface-link F in S^4 is a generalization of the definition of a spin loop on a surface-knot F_1 in S^4. It is a merit of this definition that every diffeomorphism f of S^4 sends a spin loop c on a surface-link F to a spin loop $f(c)$ on the image $f(F)$ of F in S^4. A loop-pair, a loop-pair system, or a loop basis on a surface-link F is said to be *spin* if all the loops involved in each case are spin on F.

A meridian loop (namely, the boundary loop of a meridian disk) of a handlebody smoothly embedded in S^4 is spin. For a loop-pair (k, ℓ) on F with k spin on F, one of the loop-pairs (k, ℓ) and (k, ℓ') with ℓ' a Dehn twist of ℓ along k on F is a spin loop-pair on F. Every surface-knot F_1 admits a spin loop basis, [6]. Thus, in the present definition, *every surface-link F in S^4 admits a spin loop basis.* Every (disjoint) handlebody system H smoothly embedded in S^4 is a split sum of handlebodies of H in S^4, meaning that there is a disjoint 4-ball system in S^4 such that each component

of H is contained in a 4-ball belonging to this system, and each 4-ball in this system contains exact one component of H. This can be seen by considering H as a spine graph in S^4. Every trivial surface-link F in S^4 is a split sum of trivial surface-knots in S^4 meaning that there is a disjoint 4-ball system in S^4 such that each component of F is contained in a 4-ball belonging to this system, and each 4-ball in this system contains exact one component of F. This is because F bounds a handlebody system H in S^4. Every handlebody system H smoothly embedded in S^4 is isotopic to a standard handlebody system in the standard 3-sphere S^3 with a smooth isotopy of S^4. Thus, for every handlebody system H, there is a disjoint disk pair system (D, E) smoothly embedded in S^4 with $(\partial D, \partial E) = (k, \ell)$ a spin loop basis on the trivial surface-link $\partial H = F$ such that D is a meridian disk system of H and E is disjoint from the interior of H. This disk pair system (D, E) is called a *disk basis* for the handlebody system H in S^4. More generally, a disjoint disk pair system (D, E) embedded in S^4 is called a *disk basis* for a trivial surface-link F in S^4 if $(\partial D, \partial E) = (k, \ell)$ is a spin loop basis on F and the interiors of D and E are mutually disjoint and disjoint from of F. The following *uniqueness lemma* on a disk basis for a trivial surface-link is shown.

Lemma 2.1. Let F be a trivial surface-link in S^4. For every loop basis (k, ℓ) on F, there is a disk basis (D, E) for F in S^4 with $(\partial D, \partial E) = (k, \ell)$. For every disk basis (D, E) for F, there is a handlebody system H bounded by F in S^4 such that (D, E) is a disk basis for H. For any two disk bases (D, E) and (D', E') for F, there is an orientation-preserving diffeomorphism f of S^4 keeping F setwise and sending (D, E) to (D', E'). If (D, E) is a disk basis for both handlebody systems H and H' bounded by the same F in S^4, then H' is moved into H by a smooth isotopy of S^4 keeping (D, E) fixed.

Proof. Every trivial surface-link F in S^4 admits a disk basis (D, E) for a handlebody system H bounded by F since F is moved into a split sum of standard surfaces in the standard 3-sphere S^3 by a smooth isotopy of S^4. For any spin loop bases (k, ℓ) and (k', ℓ') of a trivial surface-link F in S^4, there is an orientation-preserving diffeomorphism f of S^4 with $f'(F) = F$ such that $f'(k, \ell) = (k', \ell')$, [7]. Hence for every loop basis (k, ℓ) on F, there is a disk basis (D, E) for F in S^4 with $(\partial D, \partial E) = (k, \ell)$, which is also a disk basis for a handlebody system H bounded by F in S^4. For any two disk bases (D, E) and (D', E') for F with spin loop bases (k, ℓ) and (k', ℓ') bounded, respectively, there is an orientation-preserving diffeomorphism f' of S^4 with $f'(F) = F$ such that the disk bases $(f'(D), f'(E))$ and (D', E') for F in S^4 have the same spin loop basis (k', ℓ'). Then there is an orientation-preserving diffeomorphism f'' of S^4 with $f''(F) = F$ such that $f''(f'(D), f'(E)) = (D', E')$, [8, 9]. The composite diffeomorphism $f = f''f'$ of S^4 has $f(F) = F$ and $f(D, E) = (D', E')$. This also means that for every disk basis (D, E) for F, there is a handlebody system H bounded by F in S^4 such that (D, E) is a disk basis for H.

Assume that (D, E) is a disk basis for both handlebody systems H and H' bounded by F. Let $H \cup E \times I$ and $H' \cup E \times I$ be the 3-ball systems obtained from H and H' by adding the same 2-handle system $E \times I$ with core disk system E, respectively. Let $D \times I$ be the same 2-handle system in H and H' thickening the disk system D. Then the 3-ball systems $H \cup E \times I$ and $H' \cup E \times I$ are moved into each other by 3-cell

moves keeping the 3-ball system $D \times I \cup E \times I$ fixed. Thus, H' is moved into H by an isotopy of S^4 keeping (D, E) fixed. This completes the proof of Lemma 2.1.

Let X be a bounded smooth 3-submanifold of S^4, and H a handlebody system in S^4 with $X \cap H = \emptyset$. An *admissible arc system* between X and H is a disjoint simple arc system α whose interior is disjoint from $X \cup H$ such that every arc $\alpha_i \, (i = 1, 2, \ldots, s)$ of α joins a point p_i to a point q_i where the points $p_i \, (i = 1, 2, \ldots, s)$ are distinct points in ∂X and the points $q_i \, (i = 1, 2, \ldots, s)$ are distinct points in ∂H belonging to s distinct components of H. The following lemma shows well-definedness of a ∂-disk sum for a split sum of a bounded 3-manifold and a handlebody system.

Lemma 2.2. For any admissible arc system α between a bounded smooth 3-submanifold X and a handlebody system H in S^4 such that H is contained in the interior of a 4-ball A with $A \cap X = \emptyset$, every orientation-preserving diffeomorphism f of S^4 fixing X is X-relatively isotopic to a diffeomorphism f^* of S^4 fixing $X \cup h(\alpha) \cup H$ for a 1-handle system $h(\alpha)$ with core arc system α.

Proof. By considering the 4-ball A as a small 4-ball with $A \cap X = \emptyset$, the diffeomorphism f of S^4 is X-relatively isotopic to a diffeomorphism f' of S^4 with $f'(A) = A$. Let (D, E) be a disk basis for the handlebody system H in the interior of the 4-ball A by Lemma 2.1. Then there is an isotopic deformation of the 3-ball system $f(H \cup E \times I)$ into the 3-ball system $H \cup E \times I$ within the 4-ball A keeping ∂A fixed. Thus, the diffeomorphism f' of S^4 is X-relatively isotopic to a diffeomorphism f'' of S^4 sending the 3-ball system $H \cup E \times I$ to itself identically. By considering the 3-ball system $H \cup E \times I$ as a finite set regarding each 3-ball as a point, the admissible arc system $f''(\alpha)$ is $X \cup H$-relatively isotopic to α in S^4. Hence f is X-relatively isotopic to a diffeomorphism f^* of S^4 fixing $X \cup h(\alpha) \cup H$. This completes the proof of Lemma 2.2.

In Lemma 2.2, the 3-manifold $X \cup H \cup h(\alpha)$ is denoted by $X \#_\partial H$ and called the *∂-disk sum* of X and H in S^4 along the 1-handle system $h(\alpha)$ on $\partial X \cup \partial H$, which is determined only by the split sum $X \cup H$ with the attaching data of α and does not depend on choices of the interior of α. The 1-handle system $h(\alpha)$ is referred to as an *admissible* 1-handle system. A *connected sum* of an S^2-link K and a trivial surface-link F in S^4 is a surface-link $K \# F$ obtained from a split sum $K \cup F$ by surgery along a 1-handle system h on $K \cup F$ which is an admissible 1-handle system between a Seifert hypersurface X of K and a handlebody system H bounded by F. By Lemma 2.2, the equivalence class of the connected sum $K \# F$ is determined only by the split sum $K \cup F$ with the attaching data of the admissible core arc system α of h and does not depend on choices of the interior of α.

3. Local fusion 1-handle on boundary surface-link

Let F be a boundary surface-link, and h a local fusion 1-handle on F. Let F_h be the surface-link with two components of F to which the 1-handle h attaches, and $F_h^c = F \setminus F_h$. Let V be a disjoint Seifert hypersurface system of F, and V_h the disjoint

Seifert hypersurface system for F_h belonging to V. Let $V_h^c = V \setminus V_h$. Since h is local, take V with $h \cap V_h^c = \emptyset$. Let B be a 3-ball pair taken from each component of the two-component V_h and disjoint from h such that the intersection $dL = B \cap V_h^B$ is a disk pair in the trivial S^2-link $L = \partial B$ and a proper disk pair in V_h, where $V_h^B = \mathrm{cl}(V_h \setminus B)$. Let $V^B = V_h^B \cup V_h^c$ which is a Seifert hypersurface for the surface-link $F^B = F_h^B \cup F_h^c$ in S^4, where $F^B = \partial V^B$, $F_h^B = \partial V_h^B$ and $F_h^c = \partial V_h^c$. Then V is the ∂-disk sum $B \#_\partial V_h^B$ along an admissible 1-handle pair hdL obtained by thickening and stretching the disk pair dL. Let A be a 4-ball containing B in the interior and not meeting V_h^B whose boundary 3-sphere ∂A meets a transverse disk pair of the 1-handle pair hdL. The attaching disk pair of the fusion 1-handle h to F_h is slidden into the interior of the complementary disk pair $dL^c = \mathrm{cl}(L \setminus dL)$. Let hL be a fusion 1-handle on L obtained from h by sliding the intersection transverse disk system $h \cap V_h^B$ of h into the interior of B through V_h^B, where a thinning deformation of h is used. Further, by discarding the ∂-disk summand V_h^B, the fusion 1-handle hL on L is deformed into the interior of A. The following lemma explains some properties on a local S-trivial or S-nontrivial fusion 1-handle on a boundary surface-link.

Lemma 3.1. Let h be a local fusion 1-handle on a boundary surface-link F defined on a disjoint Seifert hypersurface system V, and h' another local fusion 1-handle on F defined on the same V and disjoint from h unless $F(h)$ is a surface-knot. Let β be a self 1-handle system on F embedded into V and disjoint from h, h' such that $V(\beta)$ is a handlebody system. The following properties (1)-(4) are obtained.

(1) The local fusion 1-handle h on F is $F(\beta) \cup B$-relatively isotopic to the fusion 1-handle hL on L in the interior of the 4-ball A which is uniquely determined by h on F up to F-relative isotopies of S^4, and independent of choices of V and β.

(2) If h is a local S-trivial fusion 1-handle on F, then h is $F(beta)$-relatively isotopic to a fusion 1-handle h_0 on F with $h_0 \cap V = h_0 \cap F$.

(3) If h is a local S-nontrivial fusion 1-handle on F, then the surface-link $F(\beta)(h)$ is equivalent to the connected sum $L(hL) \# F^B(\beta)$ along the admissible 1-handle pair hdL with $L(hL)$ a nontrivial ribbon S^2-knot in A and $F^B(\beta)$ a trivial surface-link in the 4-ball $A^c = \mathrm{cl}(S^4 \setminus A)$.

(4) Unless $F(h)$ is a surface-knot, the surface-link $F(h)$ is always a boundary surface-link and h' is a local fusion 1-handle on $F(h)$. Further, h' is S-nontrivial on F if and only if h' is S-nontrivial on $F(h)$.

Proof. To prove (1), let U be the disjoint Seifert hypersurface system of F with $h \cap U_h^c = \emptyset$, and γ a self 1-handle system on F embedded into U and disjoint from h with $U(\gamma)$ a handlebody system, used for the definition of a local S-trivial or S-nontrivial fusion 1-handle h. Assume that a boundary collar of F_h in U_h matches a boundary collar of F_h in V_h. The ∂-disk sums $B \#_\partial U_h^B$ and $B \#_\partial V_h^B$ of U_h and V_h are respectively taken along the same admissible 1-handle pair hdL. The same 4-ball A containing the 3-ball pair B but not meeting U_h^B and V_h^B is also taken. Let α be the

core arc of h. Let β_h be the subsystem of β belonging to V_h, and $\beta_h^c = \beta \setminus \beta_h$. By a thinning deformation of handlebody pair $V_h^B(\beta_h)$ keeping the meeting data with α and keeping B fixed, the handlebody pair $V_h^B(\beta_h)$ meets transversely the handlebody pair $U_h^B(\gamma_h)$ with a meridian disk system μ of $V_h^B(\beta_h)$ in the interior of $U_h^B(\gamma_h)$. Let αL^V be a simple arc spanning L obtained from α by sliding every point of the intersection point set $\alpha \cap V_h^B(\beta_h)$ into the interior of B through $V_h^B(\beta_h)$, and $\alpha^+ L$ a simple arc spanning L obtained from αL by sliding every point of the intersection point set $\alpha L \cap U_h^B$ into the interior of B through $U_h^B(\gamma_h)$ avoiding the meridian disk system μ of $V_h^B(\beta_h)$ in $U_h^B(\gamma_h)$. Let αL^U be a simple arc spanning L obtained from α by sliding every point of the intersection point set $\alpha \cap U_h^B(\gamma_h)$ into the interior of B through $U_h^B(\gamma_h)$. By these deformations, the arcs $\alpha^+ L$ and αL^V are $F(\beta) \cup B$-relatively isotopic to α since they are cell-moves of arcs, so that $\alpha^+ L$ is $F(\beta)$-relatively isotopic to αL^V in S^4. By considering the handlebody system $V^B(\beta)$ as a spine graph system, the arcs $\alpha^+ L$ and αL^V are $V(\beta)$-relatively (and hence $F(\beta) \cup B$-relatively) isotopic to L-relatively isotopic arcs in the interior of A, for $\alpha^+ L$ and αL^V are disjoint from $V^B(\beta)$ and "Regarding $V^B(\beta)$ as a spine graph"is equivalent to "Discarding $V^B(\beta)$"in this situation. Similarly, the arcs $\alpha^+ L$ and αL^U are $F(\gamma) \cup B$-relatively isotopic to L-relatively isotopic arcs in the interior of A. This means that the 1-handle hL with core arc αL^V on L in A is uniquely determined up to L-relative isotopies in A by the local fusion 1-handle h on the boundary surface-link F, independently of choices of disjoint Seifert hypersurfaces V and self 1-handle systems β disjoint from h with $V(\beta)$ handlebody systems. This completes the proof of (1).

To prove (2), assume that $F(\beta)(h)$ is a trivial surface-link. The presentation $P =< x_1, x_2 | \, x_2^{-1} w x_1 w^{-1} >$ of the fundamental group $\pi_1(A \setminus L(hL), p_0)$ of the ribbon S^2-knot $L(hL)$ in A is a presentation of an infinite cyclic group and thus there is a generator-preserving isomorphism from P to the presentation $< x_1, x_2 | \, x_2^{-1} x_1 >$, which implies that a conjugation of the word $x_2^{-1} w x_1 w^{-1}$ is reduced to a word $(x_2^{-1} x_1)^{\pm 1}$ in the free group $< x_1, x_2 >$ by the conjugacy theorem for the one relator presentation, [10]. This implies that the arc αL^V is L-relatively isotopic in A to a simple arc α_0 in A not meeting B except for the endpoints, [11]. This means that the 1-handle h is $F(\beta)$-relatively isotpic to a 1-handle h_0 with the core arc α_0 not meeting V except for the attaching part. This completes the proof of (2).

To prove (3), assume that the surface-link $F(\beta)(h)$ is a nontrivial surface-link in S^4, which means that the presentation$P =< x_1, x_2 | \, x_2^{-1} w x_1 w^{-1} >$ of the fundamental group $\pi_1(A \setminus L(hL), p_0)$ of the ribbon S^2-knot $L(hL)$ in A is a non-abelian presentation by (2), where the generators x_1, x_2 are represented by a meridian system of L in A. The surface-link $F(\beta)(h)$ is the connected sum $L(hL) \# F^B(\beta)$ of the nontrivial ribbon S^2-knot $L(hL)$ in A with fundamental group presentation P and the trivial surface-link $F^B(\beta)$ in A^c along the admissible 1-handle pair hdL, completing the proof of (3).

To prove (4), assume that $F(h)$ is not a surface-knot. The core arc α of h is deformed to transversely meet the interior of V_h with intersection number 0 by rotating near an endpoint of α along a normal circle around F_h in S^4. Then a Seifert hypersurface V_h^* for F_h with $\alpha \cap V_h^* = \alpha \cap F_h$ is obtained from V_h by surgery along a 1-handle system whose core arc system belongs to α. Then $V' = (V_h^* \cup h) \cup V_h^c$ is considered as a Seifert hypersurface system for $F(h)$ after a thinning deformation of

h. Thus, $F(h)$ is a boundary surface-link (unless it is a surface-knot). By construction, the 1-handle h' is a local fusion 1-handle on $F(h)$ on V' since h' is disjoint from h. Further, take a ∂-disk sum $B'\#_\partial {V_{h'}}^{B'}$ of V for a 3-ball pair B' taken from each component of the two-component $V_{h'}$ and disjoint from B to take a trivial S^2-link $L' = \partial B'$ in S^4 such that $dL' = B' \cap V_{h'}^{B'}$ is a disk pair in L' and a proper disk pair in $V_{h'}$. The fusion 1-handle $h'L'$ on L' obtained from h' by using V coincides with the fusion 1-handle $h''L'$ on L' obtained from h' by using V'. By (1) and (3), this means that h' is S-nontrivial on F if and only if h' is S-nontrivial on $F(h)$, completing the proof of (4). This completes the proof of Lemma 3.1.

In Lemma 3.1 (3), the fundamental group $\pi_1(A \setminus L(L^h), p_0)$ of the nontrivial ribbon S^2-knot $L(hL)$ has a non-abelian presentation $P =< x_1, x_2 | x_2^{-1} w x_1 w^{-1} >$ with generators x_1, x_2 represented by a meridian system of L in A and with a nontrivial word w given by the core arc αL of the 1-handle hL in the free group $\pi_1(A \setminus L, p_0) =< x_1, x_2 >$. This calculation is classical, [12].

4. SUPH system for ribbon surface-link and proof of Theorem 1.1

A *handled sphere system* in S^4 is a pair (O, h) embedded in S^4 such that O is a trivial S^2-link, called a *based sphere system*, and h is a 1-handle system on O. A ribbon surface-link F is a surface-link constructed from a handled sphere system (O, h) by surgery along h. This pair is uniquely constructed from a *chorded sphere system* (O, α) in S^4 with a core arc system α of h isotopically deformed into the standard 3-sphere S^3 of S^4, called a *chord system*, or from a *chord graph* (o, α) in S^3 with a trivial link o, called a *based loop system*, [3]. The chord graph (o, α) in S^3 is considered by a chord diagram $C = C(o, \alpha)$ in S^2. A *SUPH system* for a surface-link F in S^4 is a compact multi-punctured handlebody system W smoothly embedded in S^4 whose boundary ∂W is given by $\partial W = F \cup O$ for a trivial S^2-link O in S^4. The trivial S^2-link O is called the *based sphere system* for the SUPH system W. A SUPH system W for a ribbon surface-link F is constructed from a handled sphere system (O, h) by taking $W = O \times [0, 1] \cup h$ for a collar $O \times [0, 1]$ of O with $O \times \{0\} = O$ in S^4, where $O \times \{1\} = \partial W \setminus F$ is the based sphere system. Conversely, if there is a SUPH system W with based sphere system O for a surface-link F, then F is a ribbon surface-link constructed from a handled sphere system (O, h) with some 1-handle system h in W and every SUPH system W with a fixed based sphere system O for a ribbon surface-link F is unique up to boundary collars of F in W, [5]. The proof of Theorem 1.1 is done as follows.

4.1: Proof of Theorem 1.1. Let F be a boundary surface-link, and h a local S-nontrivial fusion 1-handle on F spanning the components $F_i\,(i = 1, 2)$ of F_h. By Lemma 3.1 (4), if the surface-link $F(h)$ is a ribbon surface-link then F is a ribbon surface-link, then the proof of Theorem 1.1 is completed by inductive argument. In fact, $F(h)$ is a boundary surface-link (unless it is a surface-knot) and the fusion 1-handle h'' on $F(h)$ induced from a next local S-nontrivial fusion 1-handle h' on F is a local S-nontrivial fusion 1-handle on $F(h)$. Assume that $h \cap V_h^c = \emptyset$ for a disjoint Seifert hypersurface V of F. Let B is a 3-ball pair taken from each component of

the two-component V_h and disjoint from h such that the intersection $dL = B \cap V_h^B$ is a disk pair in the trivial S^2-link $L = \partial B$ and a proper disk pair in V_h, where $V_h^B = \mathrm{cl}(V_h \setminus B)$. By Lemma 2.2, V is taken as the ∂-disk sum $B\#_\partial V^B$ along an admissible 1-handle pair hdL obtained by thickening and stretching the disk pair dL. The 3-ball pair B is taken in the interior of a 4-ball A not meeting V^B whose boundary ∂A meets hdL with a parallel copy of dL. Let β be a self 1-handle system on F embedded in V^B and disjoint from h such that $V^B(\beta)$ is a handlebody system, whose boundary is a trivial surface-link $F^B(\beta)$. The following assertion is obtained by Lemma 3.1 (1) and (3).

(4.1.1) The ∂-disk sum $W' = W(L; hL)\#_\partial V^B(\beta)$ along an admissible 1-handle pair hdL obtained by thickening and stretching the disk pair dL is a SUPH system for the surface-link $F(\beta)(h)$ such that $W(L; hL)$ is the union SUPH system $B^{(0)} \cup hL$ for the nontrivial ribbon S^2-knot $L(hL)$ in the interior of A and $V^B(\beta)$ is the handlebody system bounded by $F^B(\beta)$ in the interior of the 4-ball $A^c = \mathrm{cl}(S^4 \setminus A)$ where $B^{(0)}$ is a boundary collar of L in B and hL is a 1-handle on L constructed from h.

By assumption, the surface-link $F(h)$ is a ribbon surface-link. Let $W(F(h))$ be a SUPH system for $F(h)$. Replace $W(F(h))$ with a compact multi-punctured manifold $W(F(h))^{(0)}$ of $W(F(h))$ to avoid the intersection with β except for the attaching part of β and to avoid having an unpunctured handlebody component. Then the union $W = W(F(h))^{(0)} \cup \beta$ is a SUPH system without unpunctured handlebody components for $F(\beta)(h)$. Then the following assertion is obtained.

(4.1.2) There is an orientation-preserving diffeomorphism f of S^4 sending the SUPH system W' for $F(\beta)(h)$ to a SUPH system W_{**}^+ for $F(\beta)(h)$ obtained from some multi-fusion SUPH system W_{**} of some multi-fission SUPH system W_* of W by adding some 3-ball system B^W along some based sphere system of W_{**}.

Proof. Let $W'^{(0)}$ be a multi-punctured SUPH system of W' to avoid unpunctured handlebody components. By Theorem A.1 of Appendix, there is an orientation-preserving diffeomorphism f of S^4 sending a SUPH system $W'^{(0)}$ for $F(\beta)(h)$ to some W_{**}, which sends the SUPH system W' for $F(\beta)(h)$ to some W_{**}^+. This completes the proof of (4.1.2).

The S-nontrivial fusion 1-handle hL meets the 3-ball pair B with transverse disks by Lemma 3.1 (3). Let D_h be such a transverse disk of h, and $\partial D_h = k_h$. The proper disk $D_h^f = f(D_h)$ in W_{**}^+ is assumed to be $D_h^f \cap B^W = \emptyset$, so that D_h^f is in W_{**} and hence in W_*. It is considered that the diffeomorphism f sends the ∂-disk sum $B^{(0)}\#_\partial V^B(\beta)$ along hdL which is a SUPH system for $F(\beta)$ to the SUPH system $R = W_{**}^+(D_h^f \times I)$ for $F(\beta)$, and that h and $h^f = f(h)$ are $F(\beta)$-relatively isotopic 1-handles on $F(\beta)$ in S^4 with the same attaching disk pair in $F(\beta)$, by Lemma 3.1 (1). Let D_β be a transverse disk system of the 1-handle system β in W with just one disk for every 1-handle of β, and $\partial D_\beta = k_\beta$. The disk D_h^f has $\partial D_h^f = k_h$ disjoint from k_β and may meet the interior of the disk system D_β in W_* transversely. A proper disk system $D_h^f *$ in W_* with $D_h^f * \cap D_\beta = \emptyset$ and $\partial D_h^f * = k_h$ is constructed from D_h^f in

W_* by cut and paste operations on the transverse intersection loop system $D_h^f \cap D_\beta$ in the disk D_β. By these constructions, there is a finite sequence

$$D_h^f* = D_0 \to D_1 \to \cdots \to D_n = D_h^f$$

of proper disks D_i with $\partial D_i = k_\beta$ in W_* such that $D_h^f * \cap D_\beta = \emptyset$ and $D_{i+1} = \mathrm{cl}(D_i \setminus \delta_i) \cup \delta_i'$ $(i = 0, 1, \ldots n-1)$, where δ_i and δ_i' are disks with $\partial\delta_i = \partial\delta_i'$ such that the union $K_i = \delta_i \cup \delta_i'$ is a ribbon S^2-knot in the interior of W_* with a SUPH system U_i in W_* for every i. Let $W_* \times [0,1]$ be a collar of W_* in S^4 with $W_* \times 0 = W_*$. Let $0 = t_0 < t_1 < \ < t_n = 1$ be a division of the interval $[0,1]$. In $W_* \times [0,1]$, let $D_i[t_j] = k_h \times [0,t_j] \cup D_i \times t_j$ and $U_i[t_j] = U_i \times t_j$. Let $D_0[0] \times I = D_0 \times I$ be a 2-handle on $F(\beta)(h)$ thickening D_0 in W_*, and $D_i[t_j] \times I = k_h \times [0,t_j] \times I \cup D_i \times t_j \times I$ be a 2-handle on $F(\beta)(h)$ thickening D_i in $W_* \times [0,1]$. Let $F_i = F(\beta)(h)(D_\beta \times I)(D_i[t_i] \times I)$ $(i = 0, 1, \ldots, n)$ and $F'_{i+1} = F(\beta)(h)(D_\beta \times I)(D_i[t_{i+1}] \times I)$ $(i = 0, 1, \ldots, n-1)$. The surface-link F_0 is a ribbon surface-link since the compact 3-manifold W_*^- obtained from W_* by cutting along the disk system $D_h^f * \cup D_\beta$ is a SUPH system for F_0. Then F_1' is a ribbon surface-link since it is equivalent to F_0. The surface-link F_1 is a ribbon surface-link by Theorem A.2 of Appendix, because the based sphere system of U_i belongs to the based sphere system of W_* and F_1 is obtained from F_1' as twice SUPH moves on $U_1[t_1]$. Next, F_2' is a ribbon surface-link since it is equivalent to F_1. The surface-link F_2 is a ribbon surface-link since it is obtained from F_2' as twice SUPH moves on $U_2[t_2]$. By continuing this procedure, the surface-link $F_n = F(\beta)(h)(D_\beta \times I)(D_n[t_n] \times I) = F(\beta)(h)(D_\beta \times I)(D_h^f[1] \times I)$ is a ribbon surface-link. Since the 1-handles h and h^f on $F(\beta)$ have the same attaching disk pair in $F(\beta)$, the surface-link F_n is equivalent to $F(\beta)(h)(D_\beta \times I)(D_h \times I) = F$ by considering the 1-handle $h^f[1] = (h^f \cap F) \times [0,1] \cup h^f \times 1$ in $W_* \times [0,1]$ instead of h^f. Thus, the surface-link F is a ribbon surface-link. This completes the proof of Theorem 1.1.

Appendix: Moves on the SUPH systems of a ribbon surface-link

There are three kinds of moves M_0, M_1, M_2 on a chord diagram $C = C(o, \alpha)$ for equivalence of a ribbon surface-link F, which are called the *Reidemeister move*, the *fusion-fission move*, and the *chord move*, respectively, [3, 13]. The fusion-fission move M_1 consists of the fusion move $M_1(fusion)$ decreasing the number of based loops and the fission move $M_1(fission)$ increasing the number of based loops. For a SUPH system W' obtained from a SUPH system W by applying the Reidemeister move M_0 or the chord move M_2, there is a diffeomorphism of S^4 isotopic to to the identity and sending W into W'. Namely, W' is an isotopic deformation of W. If the fusion move $M_1(fusion)$ or its iteration produces a SUPH system W' from a SUPH system W, then W' is obtained from W by surgery of the based sphere system O of W along a fusion 1-handle or a fusion 1-handle system embedded in W, which is called a *fusion SUPH system* or a *multi-fusion SUPH system* of W, respectively. The fusion move $M_1(fusion)$ and the fission move $M_1(fission)$ are geometrically dual concepts on the SUPH systems for a ribbon surface-link. Thus, if the fission move $M_1(fission)$ or its iteration produces a SUPH system W' from a SUPH system W, then W' is obtained from W by attaching a 2-handle or a 2-handle system on the based sphere system O of W, called a *fission SUPH system* or a *multi-fission SUPH system* of W, respectively,

where note that if a 2-handle h^2 on a component O_1 of O is used for a fission SUPH system W' of the SUPH system W, then there is a 3-ball B_1 with $\partial B_1 = O_1$ in S^4 such that the intersection $d_1 = W \cap B_1$ is a 2-disk system in the interior of B_1 (coming from the 1-handle system of a handled sphere system of W) and the 2-handle h^2 on O_1 is embedded in B_1 with $h_2 \cap d_1 = \emptyset$. Any two SUPH systems W and W' for a ribbon surface-link F excluding handlebodies are moved into each other by a finite number of the moves $M_i (i = 0, 1, 2)$, [13]. Up to up to isotopic deformations, only a finite number of the fusion move $M_1(fusion)$ and/or the fission move $M_1(fission)$ remain. The following theorem is shown.

Theorem A.1. Let W and W' be SUPH systems of a ribbon surface-link F without handlebody components. Then a multi-fusion SUPH system W_{**} of a multi-fission SUPH system W_* of W up to isotopic deformations is isotopic to the SUPH system W' in S^4. In other words, a multi-fission SUPH system W_* of W up to isotopic deformations is isotopic to a multi-fission SUPH system W'_* of W' in S^4.

Proof. Assume that a SUPH system W' is obtained from a SUPH system W by $M_1(fusion)$ and a SUPH system W'' is obtained from W' by $M_1(fission)$. Then both W and W'' are obtained from W' by $M_1(fission)$. After an isotopic deformation of W' in S^4, assume that the 2-handles h^2 and $h^{2''}$ used to produce W and W'' from W' are embedded to the same 3-ball system B' bounded by the basic sphere systems O' of W' although the 1-handle system of the handled sphere system of W' may have a different position for W and W''. The 2-handles h^2 and $h^{2''}$ are made disjoint by an isotopy of B (while paying attention to the intersection between B and the 1-handle system of the handled sphere system) even if they belong to the same component of B. Then $W' \cup h^2$ is isotopically deformed to W and $W' \cup h^{2''}$ is isotopically deformed to W''. The move from $W' \cup h^2$ to $W' \cup h^2 \cup h^{2''}$ is $M_1(fission)$ and the move from $W' \cup h^2 \cup h^{2''}$ to $W' \cup h^{2''}$ is $M_1(fusion)$. This means that the ordering of the moves of $M_1(fusion)$ and $M_1(fission)$ is arranged up to isotopic deformations. Since the fusion move $M_1(fusion)$ and the fission move $M_1(fission)$ are geometrically dual concepts, a multi-fission SUPH system W_* of W up to isotopic deformations is isotopic to a multi-fission SUPH system W'_* of W'. This completes the proof of Theorem A.1.

Let F be surface-linkin S^4, and U a SUPH system for a ribbon S^2-knot K in S^4 such that $F \cap U = \emptyset$. The surface-link $F' = (F \cup K)(\eta)$ for a fusion 1-handle η on $F \cup K$ with $U \cap \eta = K \cap \eta$ a disk is called a *SUPH move* of the surface-link F by a SUPH system U. Let O^U be the based sphere system of U. The following proposition is used for the proof of Theorem 1.1.

Theorem A.2. The SUPH-move F' of a ribbon surface-link F with based sphere system O by a SUPH system U is a ribbon surface-link if the union $O \cup O^U$ is a trivial S^2-link.

Proof. The ribbon surface-link F is given by a handled sphere system (O, h). Let (O^U, h^U) be a handled sphere system for a ribbon S^2-knot K given by U, where h, h^U and η are made mutually disjoint for a fusion 1-handle η joining F to K. The

SUPH-move F' of F is obtained as the ribbon surface-link $(O \cup O^U)(h \cup h^U \cup \eta)$. This completes the proof of Theorem A.2.

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References

[1] Kawauchi, A. (2025). Ribbonness on boundary surface-link. Open Access J Phys Math 1(2), 1-4.

[2] Hosokawa, F. and Kawauchi, A. (1979). Proposals for unknotted surfaces in four-space, Osaka J Math 16, 233-248.

[3] Kawauchi, A. (2015). A chord diagram of a ribbon surface-link, J Knot Theory Ramifications 24, 1540002 (24 pages).

[4] Kawauchi, A., Shibuya, T., Suzuki, S. (1983). Descriptions on surfaces in four-space, II: Singularities and cross-sectional links, Math Sem Notes Kobe Univ 11, 31-69.

[5] Kawauchi, A. (2025). Ribbonness of a stable-ribbon surface-link, II: General case. (MDPI) Mathematics 13 (3), 402 (1-11).

[6] Hillman, J. A. and Kawauchi, A. (1995) Unknotting orientable surfaces in the 4-sphere, J Knot Theory Ramifications 4, 213-224.

[7] Hirose, S. (2002). On diffeomorphisms over surfaces trivially embedded in the 4-sphere, Algebraic and Geometric Topology 2, 791-824.

[8] Kawauchi, A. (2021). Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link, Topology and its Applications 301, 107522 (16 pages).

[9] Kawauchi, A. (2023). Uniqueness of an orthogonal 2-handle pair on a surface-link, Contemporary Mathematics (UWP) 4, 182-188.

[10] Magnus, W., Karrass, A. and Solitar, D. (1966). Combinatorial group theory: Presentations of groups in terms of generators and relations, Interscience Publishers.

[11] Marumoto, Y. (1977). Ribbon 2-knot of 1-fusion, Kobe Mathematics Seminar Notes 5 (1), 59-68.

[12] Yajima, T. (1962). On the fundamental groups of knotted 2-manifolds in the 4-space, J Math Osaka City Univ 13, 63-71.

[13] Kawauchi, A. (2018). Faithful equivalence of equivalent ribbon surface-links, J Knot Theory Ramifications 11, 1843003 (23 pages).